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Partial matrices of constant rank

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ABSTRACT

A partial matrix over a field $\mathbb{F}$ is a matrix whose entries are either elements of $\mathbb{F}$ or independent indeterminates. A completion of a partial matrix is obtained by specifying values from $\mathbb{F}$ for the indeterminates. A partial matrix of constant rank is one whose completions all have the same rank. We show that every partial matrix of constant rank r over $\mathbb{F}$ possesses an $r \times r$ partial submatrix of constant rank r if and only if $|\mathbb{F}| \geqslant r$. If $|\mathbb{F}| < r$, we show that there exist counterexamples of size $m \times n$ to this assertion provided that $\max(m, n) \geqslant r + |\mathbb{F}| - 1$.

1. Introduction

A *partial matrix* over a field $\mathbb{F}$ is a matrix whose entries are either elements of $\mathbb{F}$ or independent indeterminates. A *completion* of a partial matrix is an $\mathbb{F}$-matrix arising from an assignment of values from $\mathbb{F}$ to the indeterminate entries. We say that an $m \times n$ partial matrix has *constant rank* if each of its completions has the same rank.

In this article we investigate the following question:

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Question 1.1. Suppose that A is a partial matrix of constant rank r over a field $\mathbb{F}$. Must A possess an $r \times r$ (partial) submatrix of constant rank r?

This question obviously has an affirmative answer in the case where A is a constant matrix (i.e. has no indeterminate entries). That it also has an affirmative answer in many other cases is a consequence of a result of Huang and Zhan [6] that provides part of the motivation for this article. These authors study *affine column independent* (*ACI*) matrices, which generalize partial matrices. Entries of an ACI matrix are linear combinations of constants and indeterminates; different entries within a column may involve the same indeterminate, but indeterminates that appear in different columns are independent. The following characterization of ACI matrices of constant rank is given in [6].

Theorem 1.2. *(See Huang and Zhan [6].) Let $\mathbb{F}$ be a field with at least $\max(m, n+1)$ elements. Let A be an $m \times n$ ACI matrix over $\mathbb{F}$. Then A has constant rank r if and only if there exists a nonsingular constant matrix $T \in M_m(\mathbb{F})$ and a permutation matrix $Q \in M_n(\mathbb{F})$ such that*

$$TAQ = \begin{pmatrix} U_1 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & U_2 \end{pmatrix}$$

where $\star$ denotes an ACI matrix with no specified properties, U_1 and U_2 are square upper triangular ACI matrices with non-zero constant diagonal entries and the sum of their orders equals r.

We remark that this result of Huang and Zhan [6] can be extended to fields of order at least $\max(m, n)$ with a minor modification of their proof. A corollary of this result is that an $m \times n$ partial matrix of constant rank r over a field of order at least $\max(m, n)$ must contain an $r \times r$ partial submatrix of constant rank r.

The restriction on the field order in Theorem 1.2 is required for the mechanism of its proof in [6]. There is no discussion in [6] of cases where this condition is not satisfied. However, it is not entirely dispensable as the following example shows.

Example 1.3. This partial matrix over the field $\mathbb{F}_2$ of two elements has constant rank 3 but each of its four 3×3 submatrices admits a completion of rank 2.

$$\begin{pmatrix} 1 & 1 & x_2 & 0 \\ 1 & 0 & 0 & x_3 \\ 1 & x_1 & 1 & 1 \end{pmatrix}.$$

Problems involving ranks of completions of partial matrices have been a focus of attention in the literature for a considerable period of time. A condition for all the completions of a square partial matrix to be singular was given in 1984 by Hartfiel