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# Counterexamples to an edge spread question for zero forcing number

Gerard Jennhwa Chang [a,b,c,1], Jephian Chin-Hung Lin [c,*]

[a] *Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan*
[b] *Taida Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan*
[c] *National Center for Theoretical Sciences, Taipei Office, Taiwan*

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A B S T R A C T

This note gives counterexamples to an edge spread problem on the zero forcing number.



## 1. Introduction

For a simple graph $G$ on $n$ vertices, the *minimum rank problem* is to determine the *minimum rank* $\mathrm{mr}(G)$ of $G$, which is the smallest possible rank among all $n \times n$ symmetric

* Corresponding author.
*E-mail addresses:* gjchang@math.ntu.edu.tw (G.J. Chang), jlch3554@hotmail.com (J.C.-H. Lin).
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matrices whose off-diagonal $ij$-entry is nonzero whenever $ij$ is an edge in $G$ and zero otherwise. Equivalently, we may consider the *maximal nullity* $M(G) := n - \mathrm{mr}(G)$. In 2008, a graph parameter called the *zero forcing number* was introduced by the AIM Minimum Rank-Special Graphs Work Group [1] to serve as an upper bound for the maximal nullity.

Let $G$ be a graph with each vertex colored black or white. A *zero forcing process* on $G$ is defined by the *color-change rule*: a black vertex $x$ can force a white vertex $y$ to black at the next step when $y$ is the only white neighbor of $x$. A *zero forcing set* of $G$ is a subset $B \subseteq V(G)$ which as the initial set of black vertices can force all vertices in $V(G)$ to black. The *zero forcing number* $Z(G)$ is the minimum size of a zero forcing set of $G$. It was proved in [1] that

$$M(G) \leqslant Z(G) \quad \text{for any graph } G.$$

To describe a zero forcing process, we write $x \to y$ to denote that at some stage a black vertex $x$ forces its only white neighbor $y$ to black. The *chronological list* of a zero forcing process is a record $(x_i \to y_i\colon\ 1 \leqslant i \leqslant h)$ in order. Thus when we mention a zero forcing process, we mean the corresponding zero forcing set together with its chronological list.

A *maximal chain* is a maximal sequence of vertices of the form

$$x_1 \to x_2 \to \cdots \to x_k.$$

It is worth noting that every maximal chain is an induced path in $G$, by the color-change rule. Also, the set of all maximal chains of a zero forcing process is a collection of induced paths that cover all vertices of the graph.

The *edge spread* $z_e(G)$ of a graph $G$ on an edge $e$ is

$$z_e(G) = Z(G) - Z(G - e).$$

Edholm et al. [2] proved that if $z_e(G) = -1$, then the edge $e$ is an edge in some maximal chain of every optimal zero forcing process. Here an optimal zero forcing process means one whose corresponding zero forcing set is of size $Z(G)$. In the same paper, the authors asked in Question 2.22 if the converse of the previous statement is also true. The purpose of this note is to give a negative answer to the question by constructing infinitely many counterexamples.

## 2. The robot graphs

The counterexamples provided in this note are the *robot graphs* $R_n(h_1, h_2, \ldots, h_{2n})$ to be constructed as follows. Choose integers $n \geqslant 2$, $h_0 = 2$ and $h_s \geqslant 3$ for $1 \leqslant s \leqslant 2n$. Let $P_s$ be the path with the $h_s$ vertices labeled $p_{s,1}, p_{s,2}, \ldots, p_{s,h_s}$ for $0 \leqslant s \leqslant 2n$; and let $C_{2n+1}$ be the cycle with the $2n+1$ vertices labeled $v_0, v_1, v_2, \ldots, v_{2n}$. Define $H_p$ to be the graph obtained from $C_{2n+1}, P_0, P_1, \ldots, P_{2n}$ by identifying $p_{s,h_s}$ with $v_s$ for $0 \leqslant s \leqslant 2n$,