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Eigenvalue majorization inequalities for positive semidefinite block matrices and their blocks

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ABSTRACT

Let $H = \begin{pmatrix} M & K \\ K^* & N \end{pmatrix}$ be a positive semidefinite block matrix with square matrices M and N of the same order and denote $i = \sqrt{-1}$. The main results are the following eigenvalue majorization inequalities: for any complex number z of modulus 1,

$$\lambda(H) \prec \frac{1}{2}\lambda\big(\big[M+N+i(zK^* - \bar{z}K)\big] \oplus O\big) + \frac{1}{2}\lambda\big(\big[M+N+i(\bar{z}K - zK^*)\big] \oplus O\big).$$

If, in addition, K is Hermitian, then for any real number $r \in [-2, 2]$,

$$\lambda(H) \prec \frac{1}{2}\lambda\big((M+N+rK) \oplus O\big) + \frac{1}{2}\lambda\big((M+N-rK) \oplus O\big),$$

while if K is skew-Hermitian, then for any real number $r \in [-2, 2]$,

$$\lambda(H) \prec \frac{1}{2}\lambda\big((M+N+riK) \oplus O\big) + \frac{1}{2}\lambda\big((M+N-riK) \oplus O\big),$$

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where O is the zero matrix of compatible size. These majorization inequalities generalize some results due to Furuichi and Lin, Turkmen, Paksoy and Zhang, Lin and Wolkowicz.

1. Introduction

First, we recall the definition of majorization. Given a real vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n,$$

then we say that x is *weakly majorized* by y and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ holds, then we say that x is *majorized* by y and denote $x \prec y$. Denote by $M_{m,n}$ the set of the $m \times n$ complex matrices. $M_{n,n}$ will be abbreviated as M_n. H_n stands for the set of all Hermitian matrices of order n. Let $A \in M_n$. We always denote the singular values of $A \in M_n$ by $s_1(A) \geqslant \cdots \geqslant s_n(A)$, and we denote $s(A) := (s_1(A), \dots, s_n(A))$. Let $A \in H_n$, we always denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$ and denote $\lambda(A) := (\lambda_1(A), \dots, \lambda_n(A))$. A classical result concerning eigenvalue majorization is the fundamental result due to Schur [1,5,7,10] which states that the diagonal entries of a Hermitian matrix are majorized by its eigenvalues, i.e.,

$$\operatorname{diag}(A) \prec \lambda(A). \tag{1}$$

Ky Fan in [2] extended this result to block Hermitian matrices. Let $H = \left(\begin{smallmatrix} M & K \\ K^* & N \end{smallmatrix}\right)$ be a partitioned Hermitian matrix, where M and N are square matrices of the same order. Then

$$\lambda(M \oplus N) \prec \lambda(H). \tag{2}$$

A related result of Rotfel'd and Thompson [7] states that for positive semidefinite matrices M and N,

$$\lambda(M \oplus N) \prec \lambda\big((M + N) \oplus O\big). \tag{3}$$

In a recent article [6], Lin and Wolkowicz prove that under the conditions that H be positive semidefinite and that K be Hermitian, the eigenvalues of H are majorized by those of $M + N$, i.e.,

$$\lambda(H) \prec \lambda\big((M + N) \oplus O\big). \tag{4}$$