Linear Algebra and its Applications 446 (2014) 237–264

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the irrationality exponent of the regular paperfolding numbers ☆

Ying-Jun Guo [a], Zhi-Xiong Wen [a], Wen Wu [b,*]

[a] *Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, PR China*
[b] *Department of Mathematics, Hubei University, Wuhan, Hubei, 430062, PR China*

ARTICLE INFO

Article history:
Received 13 September 2013
Accepted 16 December 2013
Available online 7 January 2014
Submitted by R. Brualdi

MSC:
11C20
11J82
11B85

Keywords:
Paperfolding sequence
Hankel determinant
Irrationality exponent
Automatic sequence

ABSTRACT

In this paper, improving the method of Allouche et al., we calculate the Hankel determinant of the regular paperfolding sequence, and prove that the Hankel determinant sequence modulo 2 is periodic with period 10 which answers Coon's conjecture. Then we extend Bugeaud's method to obatin the exact value of the irrationality exponent for some general transcendental numbers. Using the results above, we prove that the irrationality exponents of the regular paperfolding numbers are exactly 2.

1. Introduction

Let ξ be an irrational real number, the *irrationality exponent* (or *irrationality measure*) of ξ, denoted by $\mu(\xi)$, is defined as the supremum of the set of real numbers μ such that the inequality

☆ This research is supported by NSFC (Grant Nos. 11071090, 11371156).

* Corresponding author.
E-mail addresses: guoyingjun2005@126.com (Y.-J. Guo), Zhi-xiong.wen@mail.hust.edu.cn (Z.-X. Wen), hust.wuwen@gmail.com (W. Wu).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.12.023

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

An application of the theory of continued fraction, we know that $\mu(\xi) \geqslant 2$ for all irrational number ξ. Khintchine's Theorem [18] tells us that $\mu(\xi) = 2$ for Lebesgue-almost all real numbers ξ. Furthermore, Roth's Theorem [19] asserts that $\mu(\xi) = 2$ for every algebraic irrational number.

For transcendental numbers whose continued fraction expansions are known, one can get the exact value of the irrationality exponents. For example, the irrationality exponent of e and the Sturmian numbers are given (see [1,10,16]). When we do not know the continued fraction expansions of transcendental numbers, we hardly get the exact values of their irrationality exponents. Here are two examples. One is the classical transcendental numbers $\pi, \ln(2)$ (see [20,17] and references therein), the other is the *automatic numbers* [5] defined by their expansion in some integer base. Recall that a real number ξ with b-ary expansion

$$\xi = \sum_{i \geqslant 0} u_i \frac{1}{b^i}$$

is called an automatic real number if $\{u_i\}_{i\geqslant 0}$ is an automatic sequence. In fact, all the irrational automatic real numbers are transcendental, this result was proved by Adamczewski and Bugeaud [2]. In 2006, Adamczewski and Cassaigne [3] proved that all automatic real numbers have finite irrationality exponents. In 2008, Bugeaud [9] constructed a class of real numbers whose irrationality exponent can be read off from their b-ary expansion and proved that there exist automatic real numbers with any prescribed rational irrationality exponent. In 2011, Bugeaud, Krieger and Shallit [10] showed that the irrationality exponent of every automatic (resp. morphic) number in that class is rational (resp. algebraic). And they conjectured that the result remains true for all automatic (resp. morphic) number. In 2011, applying the fact that the Hankel determinants of the Thue–Morse sequence over $\{-1, 1\}$ are nonzero [6], Bugeaud [8] proved that the irrationality exponents of the Thue–Morse real numbers are exactly 2. Using Bugeaud's method, in 2012, Coons [12] proved that the irrationality exponent of the sum of the reciprocals of the Fermat numbers is 2. Recently, Wen and Wu [21] showed that the irrationality exponents of the Cantor real numbers are exactly 2 in the same way.

In this paper, we extend Bugeaud's method to some general transcendental numbers and determine the irrationality exponent of the regular paperfolding numbers.

1.1. The main result

Let $\{u_i\}_{i\geqslant 0}$ be an integer sequence, whose generating function is $f(x) = \sum_{i\geqslant 0} u_i x^i$. The determinant of the $n \times n$-matrix $(u_{i+j-2})_{1\leqslant i,j\leqslant n}$ is called the *Hankel determinant of order n* associated with $f(x)$ (or with the sequence $\{u_i\}_{i\geqslant 0}$), denoted by $H_n(f)$. Our main result is as follows.