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The inverse, rank and product of tensors

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ABSTRACT

In this paper, we give some basic properties for the left (right) inverse, rank and product of tensors. The existence of order 2 left (right) inverses of tensors is characterized. We obtain some equalities and inequalities on the tensor rank. We also show that the rank of a uniform hypergraph is independent of the ordering of its vertices, and the Laplacian tensor and the signless Laplacian tensor have the same rank for odd-bipartite even uniform hypergraphs.

1. Introduction

For a positive integer n, let $[n] = \{1, \ldots, n\}$. An order k tensor $\mathcal{A} = (a_{i_1 \cdots i_k}) \in \mathbb{C}^{n_1 \times \cdots \times n_k}$ is a multidimensional array with $n_1 \cdots n_k$ entries, where $i_j \in [n_j]$, $j = 1, \ldots, k$.

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We sometimes write $a_{i_1 \cdots i_k}$ as $a_{i_1 \alpha}$, where $\alpha = i_2 \cdots i_k$. When $k = 2$, $\mathcal{A}$ is an $n_1 \times n_2$ matrix. If $n_1 = \cdots = n_k = n$, then $\mathcal{A}$ is an order k dimension n tensor. Recently the research on tensors has attracted extensive attention [1,2,4–6,10–13].

Now we introduce the following product of tensors.

Definition 1.1. Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$ be order $m \geqslant 2$ and $k \geqslant 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries:

$$c_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n_2]} a_{ii_2 \ldots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}},$$

where $i \in [n_1]$, $\alpha_1, \ldots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}]$.

In the above definition, if $n_1 = n_2 = \cdots = n_{k+1} = n$, then $\mathcal{AB}$ is the tensor product introduced in [4,11]. The tensor product defined in Definition 1.1 has the following properties:

(1) $(\mathcal{A}_1 + \mathcal{A}_2)\mathcal{B} = \mathcal{A}_1\mathcal{B} + \mathcal{A}_2\mathcal{B}$, where $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$, $\mathcal{B} \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$.
(2) $A(\mathcal{B}_1 + \mathcal{B}_2) = A\mathcal{B}_1 + A\mathcal{B}_2$, where $A \in \mathbb{C}^{n_1 \times n_2}$, $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$.
(3) $\mathcal{A}I_{n_2} = \mathcal{A}$, $I_{n_2}\mathcal{B} = \mathcal{B}$, where $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$, $\mathcal{B} \in \mathbb{C}^{n_2 \times \cdots \times n_{k+1}}$, I_{n_2} is the identity matrix of order n_2.
(4) $\mathcal{A}(\mathcal{BC}) = (\mathcal{AB})\mathcal{C}$, where $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$, $\mathcal{B} \in \mathbb{C}^{n_2 \times n_3 \times \cdots \times n_3}$, $\mathcal{C} \in \mathbb{C}^{n_3 \times \cdots \times n_r}$.

Clearly parts (1)–(3) follow from Definition 1.1. Part (4) will be proved at the beginning of Section 2.

The *unit tensor* of order m and dimension n is the tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ such that $\delta_{i_1 i_2 \cdots i_m} = 1$ if $i_1 = i_2 = \cdots = i_m$, and $\delta_{i_1 i_2 \cdots i_m} = 0$ otherwise.

Definition 1.2. Let $\mathcal{A}$ be a tensor of order m and dimension n, and let $\mathcal{B}$ be a tensor of order k and dimension n. If $\mathcal{AB} = \mathcal{I}$, then $\mathcal{A}$ is called an order m left inverse of $\mathcal{B}$, and $\mathcal{B}$ is called an order k right inverse of $\mathcal{A}$.

The *Segre outer product* of $a_1 \in \mathbb{C}^{n_1}, \ldots, a_k \in \mathbb{C}^{n_k}$, denoted by $a_1 \otimes \cdots \otimes a_k$, is the tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times \cdots \times n_k}$ with entries $a_{i_1 \cdots i_k} = (a_1)_{i_1} \cdots (a_k)_{i_k}$. A tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times \cdots \times n_k}$ is said to have *rank one* if there exist nonzero $a_i \in \mathbb{C}^{n_i}$ $(i = 1, \ldots, k)$ such that $\mathcal{A} = a_1 \otimes \cdots \otimes a_k$. The *rank* of a tensor $\mathcal{A}$, denoted by $\operatorname{rank}(\mathcal{A})$, is defined to be the smallest r such that $\mathcal{A}$ can be written as a sum of r rank one tensors. If $\mathcal{A} = 0$, then $\operatorname{rank}(\mathcal{A}) = 0$ (see [8]).

In this paper, some basic properties for order 2 left (right) inverse and product of tensors are given. We also obtain some results on rank of tensors and hypergraphs.