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A decreasing sequence of upper bounds for the Laplacian energy of a tree

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Let R be a nonnegative Hermitian matrix. The energy of R, denoted by $E(R)$, is the sum of absolute values of its eigenvalues. We construct an increasing sequence that converges to the Perron root of R. This sequence yields a decreasing sequence of upper bounds for $E(R)$. We then apply this result to the Laplacian energy of trees of order n, namely to the sum of the absolute values of the eigenvalues of the Laplacian matrix, shifted by $-2(n-1)/n$.

1. Notation and preliminaries

In this paper we consider undirected simple graphs. The edge set of such a graph G is denoted by $\mathcal{E}(G)$ and its vertex set by $\mathcal{V}(G)$. By an (n,m)-graph G we mean a graph

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with n vertices and m edges. The cardinality of $\mathcal{V}(G)$ i.e., n is the order of G. If G has order n, then its vertices are assumed to be labeled by $1, 2, \ldots, n$.

If $e \in \mathcal{E}(G)$ has end vertices i and j, then we say that i and j are adjacent and this edge is denoted by ij. If $i \in \mathcal{V}(G)$, then $N_G(i)$ is the set of neighbors of the vertex i in G, that is, $N_G(i) = \{j \in \mathcal{V}(G)\colon\ ij \in \mathcal{E}(G)\}$. For the i-th vertex of G, the cardinality of $N_G(i)$ is called the degree of i and it is denoted by d_i.

The adjacency matrix $A(G)$ of the graph G is a 0–1 matrix of order n with entries a_{ij}, such that $a_{ij} = 1$ if $ij \in \mathcal{E}(G)$ and $a_{ij} = 0$ otherwise. The eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A(G)$ are said to be the eigenvalues of G (see [4,6]). If G is connected, then $A(G)$ is a nonnegative irreducible matrix [4].

If $D(G)$ is the diagonal matrix of vertex degrees of G, then $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian and the signless Laplacian matrices of G, respectively (see [3,5,9,8,20]). It is well known that the spectra of $L(G)$ and $Q(G)$ coincide if and only if G is bipartite [9,8,20]. We denote by $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ and $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$ the eigenvalues of $L(G)$ and $Q(G)$, respectively.

Spectral properties of graphs, including properties of the characteristic polynomial, have been extensively studied. For details, we refer to [4].

In [10], one of the present authors recognized that the quantity

$$E(G) = E = \sum_{i=1}^{n} |\lambda_i|$$

could be viewed as a graph-spectrum-based invariant with interesting and worth-to-study mathematical properties. $E(G)$ is said to be the energy of the graph G. Details of the theory of graph energy can be found in the reviews [15,16] and the recent book [18].

In [22], Koolen and Moulton showed that the following relation holds:

$$E \leqslant \lambda_1 + \sqrt{(n-1)\left(2m - \lambda_1^2\right)}. \tag{1}$$

Then, using the inequality $2m/n \leqslant \lambda_1$ they obtained the upper bound

$$E \leqslant \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}.$$

The line graph $\mathcal{L}(G)$ of the graph G is the graph whose vertices correspond to the edges of G, with two vertices of $\mathcal{L}(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common.

Let $I(G)$ be the (vertex-edge) incidence matrix of the (n,m)-graph G, defined as the $n \times m$ matrix whose (i,j)-entry is 1 if the vertex i is incident to the edge e_j, and 0 otherwise. As well known (see [4,13]),

$$I(G)I(G)^t = A(G) + D(G) = Q(G),$$
$$I(G)^t\, I(G) = 2\, I_m + A\big(\mathcal{L}(G)\big)$$