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Classifying complements for associative algebras

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ABSTRACT

For a given extension $A \subset E$ of associative algebras we describe and classify up to an isomorphism all A-complements of E, i.e. all subalgebras X of E such that $E = A + X$ and $A \cap X = \{0\}$. Let X be a given complement and $(A, X, \triangleright, \triangleleft, \leftharpoonup, \rightharpoonup)$ the canonical matched pair associated with the factorization $E = A + X$. We introduce a new type of deformation of the algebra X by means of the given matched pair and prove that all A-complements of E are isomorphic to such a deformation of X. Several explicit examples involving the matrix algebra are provided.

0. Introduction

The concept of a *matched pair* first appeared in the group theory setting [14]. Since then, the corresponding concepts were introduced for several other categories such as Lie algebras [9], Hopf algebras [10], groupoids [5], Leibniz algebras [1], locally compact quantum groups [15], etc. With any matched pair of groups (resp. Lie algebras, Hopf

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algebras, etc.) we can associate a new group (resp. Lie algebra, Hopf algebra, etc.) called the *bicrossed product*. The bicrossed product construction is responsible for the so-called *factorization problem*, which asks for the description and classification of all objects E (groups, Lie algebras, Hopf algebras, etc.) which can be written as a 'product' of two subobjects A and X having 'minimal intersection' in E – we refer to [3] for more details, a historical background and additional references. In the setting of associative algebras, the bicrossed product was recently introduced in [4] as a special case of the more general unified product. However, in this paper we use a slightly more general construction than the one from [4], leaving aside the unitary condition on the algebras.

The classifying complements problem (CCP) was introduced in [2] in a very general, categorical setting, as a sort of converse of the factorization problem. A similar problem, called invariance under twisting, was studied in [11] for Brzezinski's crossed products. In this paper we deal with the (CCP) in the context of associative algebras:

Classifying complements problem (CCP): *Let $A \subset E$ be a given subalgebra of E. If an A-complement of E exists, describe explicitly, classify all A-complements of E and compute the cardinal of the (possibly empty) isomorphism classes of all A-complements of E (which will be called the factorization index $[E:A]^f$ of A in E).*

Another related problem which will not be discussed in this paper is that concerning the existence of complements whose natural approach is the computational one. In the sequel the existence of a complement will, however, be a priori assumed and we will be interested in describing all complements of an algebra extension $A \subset E$ in terms of one given complement.

The paper is organized as follows. In Section 1 we recall the bicrossed product for associative algebras introduced in [4]. However, the construction used in this paper is slightly more general as we drop the unitary assumption on the algebras. Section 2 contains the main results of the paper which provide the complete answer to the (CCP) for associative algebras. Let $A \subset E$ be a given extension of algebras. If X is a given A-complement of E then Theorem 2.3 provides the description of all complements of A in E: any A-complement of E is isomorphic to an r-deformation of X, as defined by (6). In other words, exactly as in the case of Hopf algebras, Lie algebras or Leibniz algebras, given X an A-complement of E all the other A-complements of E are deformations of the algebra X by certain maps $r : X \to A$ associated with the canonical matched pair which arises from the factorization $E = A + X$. The theoretical answer to the (CCP) is given in Theorem 2.6 where we explicitly construct a cohomological type object $\mathcal{HA}^2(X, A \mid (\triangleright, \triangleleft, \leftharpoonup, \rightharpoonup))$ which parameterizes all A-complements of E. We introduce the factorization index $[E:A]^f$ of a given extension $A \subset E$ as the cardinal of the (possibly empty) isomorphism classes of all A-complements. Moreover, we prove that the factorization index is computed by the formula: $[E:A]^f = |\mathcal{HA}^2(X, A \mid (\triangleright, \triangleleft, \leftharpoonup, \rightharpoonup))|$. Several explicit examples are provided. More precisely, we indicate associative algebra extensions whose factorization index is 1, 2 or 3. We end the paper with an extension of index at least 4.