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On maximum chains in the Bruhat order of $\mathcal{A}(n, 2)$

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ABSTRACT

Let $\mathcal{A}(R, S)$ denote the class of all matrices of zeros and ones with row sum vector R and column sum vector S. We introduce the notion of an inversion in a $(0, 1)$-matrix. This definition extends the standard notion of an inversion of a permutation, in the sense that both notions agree on the class of permutation matrices. We prove that the number of inversions in a $(0, 1)$-matrix is monotonic with respect to the secondary Bruhat order of the class $\mathcal{A}(R, S)$. We apply this result in establishing the maximum length of a chain in the Bruhat order of the class $\mathcal{A}(n, 2)$ of $(0, 1)$-matrices of order n in which every row and every column has a sum of 2. We give algorithmic constructions of chains of maximum length in the Bruhat order of $\mathcal{A}(n, 2)$.

1. Introduction

Let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be two vectors with nonnegative integral entries. The class of all $(0, 1)$-matrices of size m by n with row sum vector R and column sum vector S is denoted by $\mathcal{A}(R, S)$. If $m = n$ and $R = S = (k, k, \ldots, k)$, we simply write $\mathcal{A}(n, k)$ for $\mathcal{A}(R, S)$. In particular, $\mathcal{A}(n, 1)$ is the class of all permutation matrices of order n, which can be identified by the symmetric group $\mathcal{S}_n$. Combinatorial properties of the class $\mathcal{A}(R, S)$ are studied extensively (see for example [1–3,8] and the references therein).

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Given the vectors R, S and a matrix $A \in \mathcal{A}(R,S)$, one may construct a new matrix $B \in \mathcal{A}(R,S)$ from A by means of an *interchange*

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = L_2$$

that replaces a 2×2 submatrix of A equal to I_2 (if one exists) by L_2, or vice versa. In the class $\mathcal{A}(n,1)$ of permutation matrices, these interchanges correspond to transpositions in permutations. A result due to Ryser [7,8] states that for any two matrices in the class $\mathcal{A}(R,S)$, one can be obtained from the other by a sequence of $I_2 \leftrightarrow L_2$ interchanges.

In [5] Brualdi and Hwang define a *Bruhat order* on the class $\mathcal{A}(R,S)$ generalizing the classical Bruhat order on the symmetric group $\mathcal{S}_n$ (note that the class $\mathcal{A}(n,1)$ consists of permutation matrices of order n, hence it can be identified by $\mathcal{S}_n$). Given a $(0,1)$-matrix A of size m by n, let Σ_A be the m by n matrix whose (k,ℓ)-entry is

$$\sigma_{k\ell}(A) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} a_{ij}.$$

If A and C are $(0,1)$-matrices in a class $\mathcal{A}(R,S)$, then A precedes C in the Bruhat order, written as $A \preccurlyeq_B C$ for short, if $\Sigma_A \geqslant \Sigma_C$ in the entrywise order. Namely,

$$A \preccurlyeq_B C \quad \text{if and only if} \quad \sigma_{ij}(A) \geqslant \sigma_{ij}(C) \quad \text{for all } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n.$$

It is easily observed that if C is obtained from A by a sequence of $I_2 \to L_2$ interchanges, then $A \preccurlyeq_B C$. It is shown in [4] that the converse does not hold in general. This observation defines a *secondary Bruhat order* on the classes $\mathcal{A}(R,S)$ of $(0,1)$-matrices: $A \preccurlyeq_{\widehat{B}} C$ if and only if C is obtained from A by a sequence of $I_2 \to L_2$ interchanges. It is shown in [4] that the Bruhat order and the secondary Bruhat order are the same on the classes $\mathcal{A}(n,2)$, but they are different on $\mathcal{A}(6,3)$.

Answering a question asked in [4], Conflitti et al. [6] show that for all $k \geqslant 1$, the maximum length of a chain in the Bruhat order of the class $\mathcal{A}(2k,k)$ is k^4. In this work, we establish the maximum length of a chain in the Bruhat order of the classes $\mathcal{A}(n,2)$. We define the notion of an inversion in a $(0,1)$-matrix and show that the number of inversions in a $(0,1)$-matrix is monotonic with respect to the secondary Bruhat order. This result, together with a classification of minimal elements of the Bruhat order of $\mathcal{A}(n,2)$ proved in [4], gives an upper bound on the length of a chain in the (secondary) Bruhat order of the class $\mathcal{A}(n,2)$. We give algorithmic constructions of chains that achieve this upper bound.

2. Inversions in $(0,1)$-matrices

The symmetric group $\mathcal{S}_n$ is naturally identified with the class $\mathcal{A}(n,1)$ of permutation matrices of order n. In this sense, an inversion in a permutation corresponds to a pair of