Linear Algebra and its Applications 449 (2014) 28–42

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The minimization of matrix logarithms: On a fundamental property of the unitary polar factor

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ARTICLE INFO

Article history:
Received 5 August 2013
Accepted 4 February 2014
Available online 26 February 2014
Submitted by N.J. Higham

MSC:
15A16
15A18
15A24
15A44
15A45
15A60
26Dxx

Keywords:
Unitary polar factor
Matrix logarithm
Matrix exponential
Hermitian part
Minimization
Unitarily invariant norm

ABSTRACT

We show that the unitary factor U_p in the polar decomposition of a nonsingular matrix $Z = U_p H$ is a minimizer for both

$$\|\mathrm{Log}(Q^* Z)\| \quad \text{and} \quad \|\mathrm{sym}_*(\mathrm{Log}(Q^* Z))\|$$

over the unitary matrices $Q \in \mathcal{U}(n)$ for any given invertible matrix $Z \in \mathbb{C}^{n\times n}$, for any unitarily invariant norm and any n. We prove that U_p is the unique matrix with this property *to minimize all these norms simultaneously*. As important tools we use a generalized Bernstein trace inequality and the theory of majorization.

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http://dx.doi.org/10.1016/j.laa.2014.02.012
0024-3795/© 2014 Published by Elsevier Inc.

Polar decomposition
Majorization
Optimality

1. Introduction

Just as every nonzero complex number $z = re^{i\varphi}$ admits a unique polar representation with $r \in \mathbb{R}_+, \varphi \in (-\pi, \pi]$, every matrix $Z \in \mathbb{C}^{n\times n}$ can be decomposed into a product of the unitary polar factor $U_p \in U(n)$ (where $U(n)$ denotes the group of $n \times n$ unitary matrices) and a positive semidefinite matrix H [4, Lemma 2, p. 124], [19, Ch. 8], [20, p. 414]:

$$Z = U_p H.$$

This decomposition is unique if Z is invertible. We note that the polar decomposition exists for rectangular matrices $Z \in \mathbb{C}^{m\times n}$, but in this paper we shall restrict ourselves to invertible $Z \in \mathbb{C}^{n\times n}$, in which case U_p, H are unique and $H = \sqrt{Z^*Z}$ is positive definite, where the matrix square root is taken to be the principal one [19, Ch. 6].

The unitary polar factor U_p plays an important role in geometrically exact descriptions of solid materials. In this case $U_p^T F = H$ is called the right stretch tensor of the deformation gradient F and serves as a basic measure of the elastic deformation [10,29,33,28,27]. For additional applications and computational issues of the polar decomposition see e.g. [16, Ch. 12] and [26,12,24,25].

The unitary polar factor also has the property that in terms of any unitarily invariant matrix norm $\|\cdot\|$, i.e. norms that satisfy $\|X\| = \|UXV\|$ for any unitary U, V, it is the nearest unitary matrix [7, Thm. IX.7.2], [15,17], [19, p. 197] to Z, that is,

$$\min_{Q\in\mathcal{U}(n)} \|Z - Q\| = \min_{Q\in\mathcal{U}(n)} \|Q^*Z - I\| = \|U_p^*Z - I\| = \left\|\sqrt{Z^*Z} - I\right\|. \tag{1}$$

The presumably first proof – also motivated by elasticity theory – of the important case of dimension three and the Frobenius norm can be found in Grioli's work [17], see also [34].

The purpose of the present paper is to show that the unitary polar factor enjoys this minimization property (made precise in (10)) also with respect to the norm of the logarithm, an expression that arises when considering geodesic distances on matrix Lie groups (see [35,30,31] for further motivation):

$$\min_{Q\in\mathcal{U}(n)} \|\mathrm{Log}\, Q^*Z\| = \|\log U_p^*Z\| = \left\|\log \sqrt{Z^*Z}\right\|,$$

and with respect to the Hermitian part of the logarithm

$$\min_{Q\in\mathcal{U}(n)} \|\mathrm{sym}_*\, \mathrm{Log}\, Q^*Z\| = \|\mathrm{sym}_*\, \log U_p^*Z\| = \left\|\log \sqrt{Z^*Z}\right\|.$$