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Enumeration of compositions according to the sum of the values of the first letters of the occurrences of a 2-letter pattern

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ABSTRACT

A composition $\pi = \pi_1\pi_2\cdots\pi_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. The number of summands, namely m, is called the number of parts of π. We say that π contains a rise, a weak-rise, a level, a descent, or a weak-descent at position i according to whether $\pi_i < \pi_{i+1}$, $\pi_i \leqslant \pi_{i+1}$, $\pi_i = \pi_{i+1}$, $\pi_i > \pi_{i+1}$, or $\pi_i \geqslant \pi_{i+1}$. Using linear algebra, we determine formulas for generating functions that count compositions of n with m parts, according to the numbers of rises, weak-rises, levels, descents, and weak-descents, and according to the sum, over all occurrences of the rises, weak-rises, levels, descents, and weak-descents, of the first integers in their respective occurrences.

1. Introduction

A *composition* $\pi = \pi_1\pi_2\cdots\pi_m$ of a positive integer $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. (That is, π is a partition of n in which

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Table 1
Explicit formulas for the average sum of the values of the first letters of the occurrences of σ in compositions of C_n, where $\sigma \in Pat_2$.

$\sigma \in Pat_2$	Average sum of the values of the first letters of the occurrences of σ in compositions of C_n
rise	$\frac{2}{27}(3n-7)+\frac{1}{2^{n+1}}-\frac{1}{27\cdot 2^{n+1}}(6n-1)(-1)^n$, $n \geqslant 3$
descent	$\frac{1}{27}(15n-38)+\frac{3}{2^{n+1}}-\frac{1}{27\cdot 2^{n+1}}(6n+5)(-1)^n$, $n \geqslant 1$
level	$\frac{2}{27}(3n-1)+\frac{1}{27\cdot 2^{n-1}}(3n+1)(-1)^n$, $n \geqslant 1$
weak-rise	$\frac{4}{27}(3n-4)+\frac{1}{2^{n+1}}+\frac{1}{27\cdot 2^{n+1}}(6n+5)(-1)^n$, $n \geqslant 3$
weak-descent	$\frac{1}{27}(21n-40)+\frac{3}{2^{n+1}}+\frac{1}{27\cdot 2^{n+1}}(6n-1)(-1)^n$, $n \geqslant 1$

the parts are ordered.) The number m of summands is called the number of *parts* of π. For example, the compositions of the number 4 are 4, 31, 13, 22, 211, 121, 112 and 1111. We denote the set of all compositions of n or with exactly m parts, or with exactly m parts in $[d] = \{1, 2, \dots, d\}$, by C_n, or $C_{n,m}$, or $C^{[d]}_{n,m}$, respectively. Clearly, the number of compositions of n is given by $|C_n| = 2^{n-1}$.

Let $\pi = \pi_1\pi_2\cdots\pi_m$ be any composition of n with exactly m parts in $\mathbb{N}$ and let $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ be any word of length s, where $m \geqslant s$. An *occurrence* of σ in π is a subword $\pi_i\pi_{i+1}\cdots\pi_{i+s-1}$ which is order isomorphic to σ, i.e., $\pi_{i-1+a} < \pi_{i-1+b}$ if and only if $\sigma_a < \sigma_b$ for all $1 \leqslant a < b \leqslant s$. In this context, the word σ is usually called a pattern of length s (or an s-letter pattern). We denote the number of the occurrences of σ in π by $occ_\sigma(\pi)$ and we denote the sum over all occurrences of σ in π by $sfl_\sigma(\pi)$ of the values of the first letters. For example, the occurrences of the pattern 112 in the composition $\pi = 1122411$ of 12 are 112 and 224, and $sfl_{112}(\pi) = 1 + 2 = 3$.

Rises, weak-rises, levels, descents, and weak-descents can be regarded as the simplest of patterns, namely the 2-letter patterns. A *rise* corresponds to the subword pattern 12. A *weak-rise* corresponds to the subword patterns 11 and 12, a *level* to the subword pattern 11, a *descent* to the subword pattern 21, and a *weak-descent* to the subword patterns 11 and 21.

Alladi and Hoggatt [1] studied the generating function for the number of compositions of n with parts in $\{1, 2\}$, and they showed that the number of rises, levels and descents, exhibit connections with the Fibonacci sequence. Chinn and Heubach [7] generalized to parts in $\{1, k\}$, and Chinn, Grimaldi and Heubach [5] extended them to parts in $\mathbb{N}$ (for other extensions, see [6,8,9,11,12]). Later, Heubach and Mansour [10] studied the generating function for the number of compositions of n with exactly m parts according to the number of occurrences of the patterns 11, 12 and 21 (for other reference, see also [2–4] and references therein).

In this paper, for any fixed pattern $\sigma \in Pat_2 = \{Rise, Weak\text{-}rise, Level, Descent, Weak\text{-}descent\}$, we will derive the generating functions for the number of compositions, the number of parts, and the statistics occ_σ and sfl_σ. This unified framework generalizes earlier work by several authors. From this, we obtain generating functions and explicit formulas for the average sum of the values of the first letters of the occurrences of σ in compositions of n, see Table 1.