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Large regular bipartite graphs with median eigenvalue 1

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ABSTRACT

A recent result of one of the authors says that every connected subcubic bipartite graph that is not isomorphic to the Heawood graph has at least one, and in fact a positive proportion of its eigenvalues in the interval $[-1, 1]$. We construct an infinite family of connected cubic bipartite graphs which have no eigenvalues in the open interval $(-1, 1)$, thus showing that the interval $[-1, 1]$ cannot be replaced by any smaller symmetric subinterval even when allowing any finite number of exceptions. Similar examples with vertices of larger degrees are considered and it is also shown that their eigenvalue distribution has somewhat unusual properties. By taking limits of these graphs, we obtain examples of infinite vertex-transitive r-regular graphs for every $r \geqslant 3$, whose spectrum consists of points ± 1 together with intervals $[r-2, r]$ and $[-r, -r+2]$. These examples shed some light onto a question communicated by Daniel Lenz and Matthias Keller with motivation in relation to the Baum–Connes conjecture.

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1. Introduction

Let G be a graph of order n and let $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ be the eigenvalues of its adjacency matrix. In this paper we are interested in the *median eigenvalues* $\lambda_{\lfloor \frac{n+1}{2} \rfloor}(G)$ and $\lambda_{\lceil \frac{n+1}{2} \rceil}(G)$.

The eigenvalues of graphs can be useful descriptors of certain combinatorial properties of the graph. The most important one is related to the *spectral gap* (the difference $\lambda_1(G) - \lambda_2(G)$), which certifies expansion properties of graphs, see, e.g. [1,10,6]. A more recent application comes from mathematical chemistry [3,4,7] in relation to the HOMO–LUMO energies of molecular graphs. In this setting, the median eigenvalues play the central role. Motivated by questions in mathematical chemistry, one of the authors proved the following curious result.

1.1. Theorem. *(Mohar [8]) Let G be a bipartite subcubic graph. If every connected component of G is isomorphic to the Heawood graph, then its median eigenvalues are $\pm\sqrt{2}$. In any other case, the median eigenvalues lie in the interval $[-1, 1]$.*

In fact, this theorem can be strengthened.

1.2. Theorem. *(Mohar [8]) There is a constant $\delta > 0$ such that for every bipartite subcubic graph G of order n, none of whose connected components is isomorphic to the Heawood graph, at least $\lceil \delta n \rceil$ of its eigenvalues belong to the interval $[-1, 1]$.*

However, the paper [8] leaves an open question:

> *Is there a strengthening of Theorem 1.1 where the interval $[-1, 1]$ is replaced by a smaller symmetric interval around 0 if we allow a finite number of exceptional graphs?*

In this note we answer the question in the negative by proving:

1.3. Theorem. *There are infinitely many connected cubic bipartite graphs that have no eigenvalues in the open interval $(-1, 1)$.*

By Theorem 1.2, large graphs in the family of Theorem 1.3 will have ± 1 as eigenvalues of large multiplicity.

The construction of graphs used to prove Theorem 1.3 can be generalized to larger vertex degrees, providing examples with unusual eigenvalue distributions.

1.4. Theorem. *For every integer $r \geqslant 3$, there are infinitely many connected r-regular bipartite graphs with median eigenvalues ± 1 but with no eigenvalues in the intervals $(-1, 1)$ and $\pm(1, r - 2)$.*