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Distribution of the discretization and algebraic error in numerical solution of partial differential equations

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A B S T R A C T

In the adaptive numerical solution of partial differential equations, local mesh refinement is used together with a posteriori error analysis in order to equilibrate the discretization error distribution over the domain. Since the discretized algebraic problems are *not solved exactly*, a natural question is whether the spatial distribution of the algebraic error is analogous to the spatial distribution of the discretization error. The main goal of this paper is to illustrate using standard boundary value model problems that this may not hold. On the contrary, the algebraic error can have large local components which can significantly dominate the total error in some parts of the domain. The illustrated phenomenon is of general significance and it is not restricted to some particular problems or dimensions. To our knowledge, the discrepancy between the spatial

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Adaptivity
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Spatial distribution of the error

distribution of the discretization and algebraic errors has not been studied in detail elsewhere.

1. Introduction

In numerical solution of partial differential equations, a sufficiently accurate solution (the meaning depends on the particular problem) of the linear algebraic system arising from discretization has to be considered. When the finite element method (FEM) is used for discretization, the system matrix is sparse. The sparsity of the algebraic system matrix is presented as a fundamental advantage of the FEM. It allows to obtain a numerical solution when the problem is hard and the discretized linear system is very large. It is worth, however, to examine some *mathematical* consequences which do not seem to be addressed in the FEM literature.

The FEM generates an approximate solution in form of a linear combination of basis functions with *local* supports. Each basis function multiplied by the proper coefficient thus approximates the desired solution only locally. The *global* approximation property of the FEM discrete solution is then ensured by solving the linear algebraic system for the unknown coefficients; the linear algebraic system links the local approximation of the unknown function in different parts of the domain. If the linear algebraic system is solved *exactly*, then all is fine. But in practice we do not solve exactly. In hard problems we even *do not want* to achieve a small algebraic error. That might be too costly or even impossible to get; see, e.g., [7, Sections 1–3], [24, Sections 1 and 6], [33, Section 2.6], the discussion in [34, pp. 36 and 72], and [38, Section 1]. Then, however, one should naturally ask whether the spatial distribution of the algebraic error in the domain can significantly differ from the distribution of the discretization error. There is no a priori evidence that these distributions are to be analogous. On the contrary, from the nature of algebraic solvers, either direct or iterative, there seems to be no reason for equilibrating the algebraic error over the domain. Numerical results presented in this paper demonstrate that the algebraic error can indeed significantly dominate the total error in some part of the domain. To our knowledge, apart from a brief discussion in [26, Sections 5.1 and 5.9.4], the presented phenomenon has not been studied elsewhere.

In order to avoid misunderstandings, it is worth to point out that the phenomenon described in this paper is not related to the so-called "smoothing properties" of the conjugate gradient (CG) method [23] or to the investigation of smoothing in the multilevel setting (for such analyses see, e.g., [36] or [41, Chapter 9]). Moreover, it is *not* due to the particular iterative solver or due to the specifics of the model problems used in this paper for illustration. Following the standard methodology used in the numerical PDE literature for decades (see, e.g., [8,15,19]), we start by illustrating the phenomenon using the simplest 1D boundary value problem. Furthermore, in order to plot illustrative figures, we use a small number of discretization nodes. In order to avoid the impression