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A note on Laplacian eigenvalues and domination

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ABSTRACT

In 2005, Lu, Liu and Tian [2] published two inequalities involving the domination number and Laplacian eigenvalues of graphs. A short proof of their results is given, an error is pointed out, and improved inequalities are presented.

For a graph $G = (V, E)$, let $L(G) \in \mathbb{R}^{V \times V}$ be its Laplacian matrix. For $i = 1, \ldots, |V|$, let $\mu_i(G)$ be the ith largest eigenvalue of $L(G)$ with multiplicities accounted for. Since the Laplacian matrix is positive semidefinite, $\mu_i(G) \geqslant 0$ for every i. We refer to [3] for more details about Laplacian matrices and their eigenvalues.

The largest eigenvalue $\mu_1(G)$ is also called the *Laplacian spectral radius* of G. For every edge e of G, the difference of Laplacian matrices $L(G) - L(G - e)$ is a positive semidefinite matrix. This implies that the Laplacian spectral radius is *monotone*, i.e., for every subgraph H of G, $\mu_1(H) \leqslant \mu_1(G)$.

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A set D of vertices of G is a *dominating set* if every vertex is either in D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in the graph. It is NP-hard to determine the domination number of a graph [1], thus it is of interest to find nontrivial bounds on this parameter.

The paper [2] by Lu, Liu, and Tian has two main results that are listed as statements (1) and (2) below.

(1) (Theorem 8 in [2]) If G is a connected graph of order $n \geqslant 2$, then

$$\mu_{n-1}(G) \leqslant n - \frac{n}{n-\gamma(G)}\big(\gamma(G)-1\big).$$

The bound in (1) corresponds to Theorem 8 in [2], which has an addendum that equality holds if and only if G is isomorphic to the cycle of length 4. However, that claim is false since K_2 is another example, for which equality holds.

We give a short proof of an improved inequality below.

Theorem 1. *If G is a graph of order n that has no isolated vertices, then*

$$\mu_{n-1}(G) \leqslant n - 2\big(\gamma(G)-1\big).$$

Proof. Let $k = \gamma(G)$. It is easy to see that G has a minimum dominating set $D = \{x_1, \dots, x_k\}$ such that for every vertex $x_i \in D$, there is a vertex $x_i' \in V(G) \setminus D$ whose only neighbor in D is x_i. (This fact is well known; a proof also appears in [2] as Lemma 2.) Let H be the complement of G. The subgraph of H induced on vertices $\{x_i, x_i' \mid 1 \leqslant i \leqslant k\}$ contains as a subgraph the graph H' that is isomorphic to $K_{k,k}$ with a perfect matching removed. Observe that this is a $(k-1)$-regular bipartite graph and thus its Laplacian spectral radius is $2(k-1)$. By the monotonicity of the Laplacian spectral radius, we conclude that $\mu_1(H) \geqslant \mu_1(H') = 2(k-1)$. On the other hand, eigenvalues of complementary graphs are related: $\mu_{n-1}(G) = n - \mu_1(H)$ (see [3]). This gives the inequality of the theorem. □

Note that every graph without isolated vertices satisfies that $\gamma(G) \leqslant n/2$, and hence $n/(n-\gamma(G)) \leqslant 2$. This shows that the inequality in Theorem 1 is an improvement over the bound in (1) and coincides with it if and only if $\gamma(G) = n/2$.

There are many instances where the inequality of Theorem 1 is tight. It follows from the proof above that this happens if and only if $\mu_1(H) = 2(k-1)$ (where we use the notation from the proof). If H contains an edge joining a vertex in H' with a vertex v that is not in H', then it is easy to see that $\mu_1(H) > 2(k-1)$. (To see this, consider the vector whose values are 1 on the vertices x_i and -1 on x_i' for $i = 1, \dots, k$, and has value 0 on v, compute the Rayleigh quotient for the Laplacian matrix of this graph, and then use monotonicity.) If such a vertex does not exist, then every vertex in $V(G) \setminus V(H')$ is adjacent in G to all vertices in $V(H')$. It follows that if G has a vertex z that is not