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Integrally normalizable matrices and zero–nonzero patterns

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A B S T R A C T

The problem of determining whether or not an $n \times n$ integer matrix is diagonally similar to an integer matrix with $n - 1$ off-diagonal entries equal to 1 is studied. Such a matrix is called integrally normalizable, and a zero–nonzero pattern is integrally normalizable if each matrix with this zero–nonzero pattern is integrally normalizable with respect to the same set of $n - 1$ off-diagonal entries. Matrices that are integrally normalizable with respect to a fixed spanning tree, and integrally normalizable zero–nonzero patterns are characterized. The maximum number of nonzero entries in an $n \times n$ integrally normalizable zero–nonzero pattern is shown to be $\frac{n^2+3n-2}{2}$. Extensions to matrices over other integral domains are also presented.

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1. Introduction

It is well known that diagonal similarity is useful for reducing the number of variables in spectral problems involving zero–nonzero and sign patterns; see, e.g., [1–5]. In these problems the entries corresponding to a spanning tree of the digraph associated with the pattern can be assumed to be ± 1.

We are interested in diagonal similarity of matrices over an integral domain, in particular, in determining whether or not an $n \times n$ matrix over the integral domain is diagonally similar (by a matrix with entries in its field of fractions) to another matrix with $n-1$ off-diagonal entries equal to 1 and all entries in the integral domain. Furthermore, given an $n \times n$ zero–nonzero pattern, we are interested in determining whether or not *all* matrices over an integral domain with this zero–nonzero pattern are diagonally similar to such a matrix with the same $n-1$ off-diagonal entries equal to 1.

For convenience we restrict consideration to the ring of integers $\mathbb{Z}$, except in Section 7 in which we consider the integral domain $\mathbb{R}[x_1, x_2, \ldots, x_n]$. An important example in which the above diagonal similarity is possible is the case of proper lower Hessenberg patterns in which each entry on the superdiagonal can be normalized to 1. To describe this case (see Example 4.2 below) and the problem in general we provide the following definitions.

A *zero–nonzero pattern* $\mathcal{A} = [\alpha_{ij}]$ is a matrix with entries in $\{0, *\}$. An *integer realization* of $\mathcal{A}$ is a matrix $A = [a_{ij}]$ for which $a_{ij} = 0$ if $\alpha_{ij} = 0$, and a_{ij} is a nonzero integer if $\alpha_{ij} = *$. The set of all integer realizations of $\mathcal{A}$ is denoted by $Q_{\mathbb{Z}}(\mathcal{A})$.

Definition 1.1. An $n \times n$ integer matrix A is *integrally normalizable* if there exists an $n \times n$ (rational) nonsingular diagonal matrix D such that $B = DAD^{-1}$ is an integer matrix with at least $n-1$ off-diagonal entries equal to 1. An $n \times n$ zero–nonzero pattern $\mathcal{A}$ is *integrally normalizable* if each matrix $A \in Q_{\mathbb{Z}}(\mathcal{A})$ is integrally normalizable with respect to the same set of $n-1$ off-diagonal entries equal to 1.

Note that the diagonal entries of A are irrelevant for deciding whether or not A is integrally normalizable since these are preserved under diagonal similarity.

We focus on formulating conditions for a matrix to be integrally normalizable and on characterizing zero–nonzero patterns that are integrally normalizable. The following example illustrates these ideas.

Example 1.2. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 12 \\ 11 & 0 & -3 \\ 5 & -7 & 0 \end{bmatrix}.$$