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Inequalities with determinants of perturbed positive matrices

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ABSTRACT

We prove two inequalities regarding the ratio $\det(A+D)/\det A$ of the determinant of a positive-definite matrix A and the determinant of its perturbation $A+D$. In the first problem, we study the perturbations that happen when positive matrices are added to diagonal blocks of the original matrix. In the second problem, the perturbations are added to the inverses of the matrices.

1. Introduction

Given k complex square matrices $B_1, \dots, B_k$ of format $n_1 \times n_1, n_2 \times n_2, \dots, n_k \times n_k$, let us denote by $\operatorname{diag}(B_1, \dots, B_k)$ the matrix of the format $(n_1 + \dots + n_k) \times (n_1 + \dots + n_k)$ whose main diagonal blocks are $B_1, \dots, B_k$ and all other entries are 0. In other words:

$$\operatorname{diag}(B_1, \dots, B_k) = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & B_k \end{bmatrix}.$$

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Given two vectors $u, v \in \mathbb{C}^k$ such that $u = \langle u_1, \ldots, u_k \rangle$ and $v = \langle v_1, \ldots, v_k \rangle$, we define their inner product $\langle u, v \rangle = \sum_{i=1}^{k} u_i \overline{v_i}$. For a complex $n \times m$ matrix R, we use R^* to denote its adjoint matrix. In other words, R^* is the transpose of the complex conjugate of R, and for $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$ the following is satisfied: $\langle Ru, v \rangle = \langle u, R^* v \rangle$. A square matrix A is self-adjoint if $A^* = A$.

The self-adjoint matrix A of format $n \times n$ is called positive (or positive definite) if $\langle Ax, x \rangle > 0$ for each non-zero vector $x \in \mathbb{C}^n$. If the strict inequality is replaced by $\geqslant$, the matrix is called non-negative (or positive semi-definite). If A and B are two square matrices of the same format, we will write $A \geqslant B$ (resp. $A > B$) if $A - B \geqslant 0$ (resp. $A - B > 0$).

For $n \in \mathbb{N}$ we will denote by I_n the $n \times n$ identity matrix. The subscript n will be omitted when there is no danger of ambiguity.

We will prove the following two inequalities regarding positive matrices with complex entries.

Theorem 1.1. *Assume that $k \in \mathbb{N}$ and that $n_1, \ldots, n_k$ are positive integers. Assume that $(C_i)_{i=1}^k$ and $(D_i)_{i=1}^k$ are two sequences of positive matrices such that for each $i \in \{1, 2, \ldots, k\}$ the matrices C_i and D_i are of format $n_i \times n_i$. Assume that C is a positive matrix whose diagonal blocks are $C_1, \ldots, C_k$. The following inequality holds:*

$$\frac{\det(C + \operatorname{diag}(D_1, \ldots, D_k))}{\det C} \geqslant \frac{\det(C_1 + D_1)}{\det C_1} \cdots \frac{\det(C_k + D_k)}{\det C_k}. \tag{1}$$

Theorem 1.2. *Assume that $k \in \mathbb{N}$ and that $n_1, \ldots, n_k$ are positive integers. Assume that $(C_i)_{i=1}^k$ and $(D_i)_{i=1}^k$ are two sequences of positive matrices such that for each $i \in \{1, 2, \ldots, k\}$ the matrices C_i and D_i are of format $n_i \times n_i$. Assume that C is a positive matrix such that the diagonal blocks of C^{-1} are $C_1^{-1}, \ldots, C_k^{-1}$. The following inequality holds:*

$$\frac{\det(C + \operatorname{diag}(D_1, \ldots, D_k))}{\det C} \leqslant \frac{\det(C_1 + D_1)}{\det C_1} \cdots \frac{\det(C_k + D_k)}{\det C_k}. \tag{2}$$

The two inequalities presented in this paper have the flavor of Fischer's determinantal inequality, although in (1) the sign is reversed. An inequality related to our results, which features quotients of perturbed matrices, has been established previously [7]. For refinements of Fischer-type inequalities with singular values, the reader is referred to [3] and [4]. After taking the logarithms of left and right sides of the inequality (2), one obtains

$$\varphi\big(C, \operatorname{diag}(D_1, D_2)\big) \leqslant \varphi(C_1, D_1) + \varphi(C_2, D_2),$$

where $\varphi(X, Y) = \log\det(X + Y) - \log\det(X)$. Similar inequalities are known to hold for concave functions φ, and such results can be found in [1].