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Sharp bounds for the spectral radius of nonnegative matrices

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We give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with all row sums positive using its average 2-row sums, and characterize the equality cases if the matrix is irreducible. We compare these bounds with the known bounds using the row sums by examples. We also apply these bounds to various matrices associated with a graph, including the adjacency matrix, the signless Laplacian matrix and some distance-based matrices. Some known results are generalized and improved.

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1. Introduction

Let A be an $n \times n$ nonnegative matrix. The spectral radius (alias Perron root) of A, denoted by $\rho(A)$, is the largest modulus of eigenvalues of A. See [2,8,13,16,18,21,22] for some known properties of the spectral radius of nonnegative matrices.

In this paper, we also consider the spectral radius of some nonnegative matrices associated with a graph. Let G be a simple undirected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$.

The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise [5]. For $1 \leqslant i \leqslant n$, let d_i be the degree of vertex v_i in G. Let $Deg(G)$ be the degree diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$. The signless Laplacian matrix of G is the $n \times n$ matrix $Q(G) = Deg(G) + A(G)$ [7]. The spectral radius of the adjacency matrix has been studied extensively (see, e.g., [6,8,12,15,19]), and the spectral radius of the signless Laplacian matrix has also received much attention (see, e.g., [8,11,20,23]).

Suppose that G is connected. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_{ij})$, where d_{ij} is the distance between vertices v_i and v_j, i.e., the number of edges of a shortest path connecting them, in G [9,14]. For $1 \leqslant i \leqslant n$, the transmission D_i of vertex v_i in G is the sum of distances between v_i and (other) vertices of G. Let $Tr(G)$ be the transmission diagonal matrix $\mathrm{diag}(D_1, \ldots, D_n)$. The distance signless Laplacian matrix of G is the $n \times n$ matrix $\mathcal{Q}(G) = Tr(G) + D(G)$ [1]. The reciprocal distance matrix (alias Harary matrix) of G is the $n \times n$ matrix $R(G) = (r_{ij})$, where $r_{ij} = \frac{1}{d_{ij}}$ for $i \neq j$, and $r_{ii} = 0$ for $1 \leqslant i \leqslant n$ [14]. Some results have been obtained for the spectral radius of these distance-based matrices of a connected graph (see, e.g., [8,24]).

Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. For $1 \leqslant i \leqslant n$, the i-th row sum of A is $r_i(A) = \sum_{j=1}^{n} a_{ij}$. Duan and Zhou [8] found upper and lower bounds for the spectral radius of a nonnegative matrix using its row sums, and characterized the equality cases if the matrix is irreducible. They also applied those bounds to the nonnegative matrices associated with a graph as mentioned above.

For $1 \leqslant i \leqslant n$ and an $n \times n$ nonnegative matrix $A = (a_{ij})$ with $r_i(A) > 0$, the i-th average 2-row sum of A is defined as $m_i(A) = \frac{\sum_{k=1}^{n} a_{ik} r_k(A)}{r_i(A)}$. For a graph G on n vertices with $d_i > 0$, $m_i(A(G)) = \frac{\sum_{v_i v_j \in E(G)} d_j}{d_i}$, which is known as the average 2-degree of vertex v_i in G [3,17]. Huang and Weng [10] gave an upper bound for the spectral radius of the adjacency matrix of a connected graph with at least two vertices using its average 2-degrees (cf. Chen et al. [4]).

In this paper, we give sharp upper and lower bounds for the spectral radius of a nonnegative matrix with all row sums positive using the average 2-row sums, and characterize the equality cases if the matrix is irreducible. Then we compare these bounds with those using the row sums presented in [8] by examples. We also apply these results to various matrices associated with a graph as mentioned above. Some known results are generalized and improved.