Linear Algebra and its Applications 449 (2014) 381–401

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Structure of the rational monoid algebra for Boolean matrices of order 3

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ARTICLE INFO

Article history:
Received 8 July 2013
Accepted 18 February 2014
Available online 13 March 2014
Submitted by R. Brualdi

MSC:
primary 15B34
secondary 16-04, 16G10, 16N99, 16Z05, 20M30, 68W30

Keywords:
Boolean matrices
Associative algebras
Radical
LLL algorithm
Semisimple quotient
Wedderburn decomposition
Simple two-sided ideals
Rational representations

ABSTRACT

We use the computer algebra system `Maple` to study the 512-dimensional associative algebra $\mathbb{Q}\mathcal{B}_3$, the rational monoid algebra of 3×3 Boolean matrices. Using the LLL algorithm for lattice basis reduction, we obtain a basis for the radical in bijection with the 42 non-regular elements of $\mathcal{B}_3$. The center of the 470-dimensional semisimple quotient has dimension 14; we use a splitting algorithm to find a basis of orthogonal primitive idempotents. We show that the semisimple quotient is the direct sum of simple two-sided ideals isomorphic to matrix algebras $M_d(\mathbb{Q})$ for $d = 1, 1, 1, 2, 3, 3, 3, 3, 6, 6, 7, 9, 9, 12$. We construct the irreducible representations of $\mathcal{B}_3$ over $\mathbb{Q}$ by calculating the representation matrices for a minimal set of generators.

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[1] This research was partially supported by a Discovery Grant from NSERC.
[2] The author thanks the referee for valuable comments.

http://dx.doi.org/10.1016/j.laa.2014.02.035
0024-3795/

1. Introduction

We write $\mathcal{B}_n$ for the monoid of $n \times n$ Boolean matrices with the usual Boolean matrix product $(1+1=1)$; equivalently, $\mathcal{B}_n$ is the monoid of binary relations on n elements with the relative product:

$$R \circ S = \{(i,j) \mid \text{for some } k \text{ we have } (i,k) \in R \text{ and } (k,j) \in S\}.$$

Konieczny [15] has recently published a complete proof of Devadze's theorem [8] on minimal sets of generators for $\mathcal{B}_n$.

The representation theory of $\mathcal{B}_n$ over the field $\mathbb{Q}$ of rational numbers has received little attention. One reason for this is Preston's theorem [16], which states that any finite group is a maximal subgroup of $\mathcal{B}_n$ for some n; it is therefore not realistic to expect a uniform structure theory for the rational monoid algebra $\mathbb{Q}\mathcal{B}_n$. (The simplest non-trivial case $n=2$ appears in [5].)

This paper studies the irreducible rational representations of $\mathcal{B}_n$ for $n=3$ by determining the structure of the rational monoid algebra $\mathbb{Q}\mathcal{B}_3$. We use the computer algebra system Maple, together with a constructive approach to the classical structure theory of finite dimensional associative algebras; see [3]. Our use of the LLL algorithm for lattice basis reduction [4] (with various values of the reduction parameter) can be considered as a somewhat novel heuristic not present in the standard algorithms. This has allowed us to compute a particularly convenient basis for the radical, in bijection with the non-regular elements of $\mathcal{B}_3$. Our computations reveal some remarkable features of $\mathbb{Q}\mathcal{B}_3$: the center of the semisimple quotient splits completely over the rationals, and even the simple components are completely split. It is an open problem to determine whether these properties hold only for $\mathcal{B}_3$ or if they remain valid for larger Boolean matrices.

We summarize the contents of the paper. Section 2 recalls some basic results about the structure of the monoid $\mathbf{A} = \mathcal{B}_3$. Section 3 determines the radical $\mathbf{R} \subset \mathbf{A}$; we find that its dimension is 42 and we use the LLL algorithm to find a natural basis of $\mathbf{R}$ in bijection with the non-regular elements of $\mathcal{B}_3$. Section 4 determines the structure constants for the semisimple quotient $\mathbf{S} = \mathbf{A}/\mathbf{R}$, and a basis for the 14-dimensional center $\mathbf{C} \subset \mathbf{S}$. We then apply the splitting algorithm of Ivanyos and Rónyai [11] to determine a new basis for $\mathbf{C}$ consisting of orthogonal primitive idempotents. Section 5 determines the decomposition of $\mathbf{S}$ into a direct sum of simple two-sided ideals; in particular, we find that $\mathbb{Q}$ is the splitting field of $\mathcal{B}_3$. The dimensions of the irreducible representations are $d = 1,1,1,2,3,3,3,3,6,6,7,9,9,12$. We then find a minimal left ideal in each simple two-sided ideal, and construct an isomorphism of each simple two-sided ideal with the matrix algebra $M_d(\mathbb{Q})$. Section 6 calculates the representation matrices for the minimal set of five generators of $\mathcal{B}_3$ obtained from Devadze's theorem [8,15]; the matrix entries belong to $\{0,1,-1\}$, and only the four representations of dimensions $d \geqslant 7$ are faithful. Section 7 gives some suggestions for further research.