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Generalized inverses: Uniqueness proofs and three new classes

Michael P. Drazin

Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067, USA

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ABSTRACT

Given any ring R with 1 and any $a,b,c \in R$, then, generalizing ideas of J.J. Koliha and P. Patrício in 2002 and of Z. Wang and J. Chen in 2012, a is called "(b,c)-pseudo-polar" if there exists an idempotent $p \in R$ such that $1-p \in (bR+J) \cap (Rc+J)$, pb and $cp \in J$ (where J denotes the Jacobson radical of R) and p lies in the second commutant of a. This p is shown to be unique whenever it exists. A new outer generalized inverse y of a, called the (b,c)-pseudo-inverse of a, is also defined, and the existence of y is shown to imply that a is (b,c)-pseudo-polar, and hence that y is itself unique. Generalizing results of Koliha, Patrício, Wang and Chen, further connections between the (b,c)-pseudo-polar and (b,c)-pseudo-invertible properties are found, and the (b,c)-pseudo-invertibility of a_1a_2 is shown to imply a corresponding property for a_2a_1. Two further types of uniquely-defined outer generalized inverses are also introduced.

E-mail address: mdrazin@math.purdue.edu.

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1. Introduction

In 2012 a new class of generalized "(b,c)-inverses" y was introduced in [8], defined for suitable elements a of arbitrary multiplicative semigroups S. This class includes as special cases essentially all of the "first generation" of individual, specific, well-defined (outer) generalized inverses, i.e. those discovered before (say) 1980.

It is perhaps surprising that, after so much effort by so many writers, to this day apparently only two genuinely different arguments for the uniqueness of a generalized inverse y of $a \in S$ have been discovered. Among the earliest generalized inverses, such as the Moore–Penrose inverse $y = a^\dagger$ and the present author's pseudo-inverse $y = a'$ (now usually called the Drazin inverse a^D), their respective proofs of uniqueness at first sight appear to be merely very similar. However, at a higher level of generality, Ref. [8, p. 1912, Theorem 2.1] now shows that in fact the proofs for $a^\dagger$ and a^D (as well as those for essentially all other pre-1980 generalized inverses) are direct special cases of a single more general unified argument valid for the much larger class of (b,c)-inverses. Moreover, all these arguments are only slight elaborations of the usual one-line proof $x = \cdots = xay = \cdots = y$ for the uniqueness of the ordinary classical inverse a^{-1} in semigroups or rings.

Thus any prospects of discovering new generalized inverses may be linked to finding different proofs of uniqueness, and it is only quite recently that a second approach has emerged, namely to assume first that $ay = ya$ and $yay = y$, and then, by the use of appropriate further axioms (in any associative ring R with 1), to establish the uniqueness of y via that of $p = 1 - ay = 1 - ya$. (For a more precise description of what this involves, see Section 3.)

The first authors to adopt this approach were J.J. Koliha and P. Patrício [12, p. 138, Definition 2.2] in 2002 (see also [11]), who defined $a \in R$ as being *quasipolar* if there exists $p \in R$ such that

(1) $p^2 = p \in \text{comm}^2(a)$,
(2) $a + p$ is a two-sided unit of R, and
(3) $1 + xap$ is a two-sided unit of R for every $x \in R$ satisfying $apx = xap$,

where $\text{comm}^2(a)$ in (1) is defined as usual by

$$\text{comm}^2(a) = \{r : r \in R \text{ and } ad = da \text{ for } d \in R \text{ implies } rd = dr\}.$$

Koliha and Patrício showed [12, p. 139, Proposition 2.3] that under these conditions p is unique, and that this p leads to a corresponding unique generalized inverse y for a [12, p. 141, Theorem 4.2], which they call the *g-Drazin inverse* of a.

Building on these ideas, in 2012 Z. Wang and J. Chen introduced [17, p. 1333, Definition 1.1] another quite similar property, calling $a \in R$ *pseudopolar* if there exists $p \in R$ satisfying (1), (2) and

(4) $a^k p \in J$ for some $k \in \mathbb{N}$,