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# Idempotent zero patterns ☆

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ABSTRACT

We first characterize idempotent zero patterns. Then we determine the possible numbers of nonzero entries in an idempotent zero pattern with a given minimum rank and characterize those patterns that attain the extremal numbers. The results can be stated in terms of 0–1 matrices.



## 1. Introduction

A matrix with entries from the set $\{0, *\}$ is called a *zero pattern*. We denote by $M_{n,k}(F)$ the set of $n \times k$ matrices over a field $F$. Given a field $F$ and an $n \times k$ zero pattern $A = (a_{ij})$, we denote by $Z_F(A)$ the set of $n \times k$ matrices over $F$ with zero pattern $A$; i.e.,

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$$Z_F(A) = \{B = (b_{ij}) \in M_{n,k}(F) \mid b_{ij} = 0 \text{ if and only if } a_{ij} = 0\}.$$

Thus $*$ indicates nonzero entries.

At a seminar last year, Professor Xingzhi Zhan posed the following problem.

**Problem 1.** Given a field $F$, characterize those square zero patterns $A$ such that $B^2 \in Z_F(A)$ whenever $B \in Z_F(A)$.

For a given field $F$, we call a zero pattern which satisfies the condition in Problem 1 an *idempotent zero pattern* over $F$. We denote by $\Omega_n(F)$ the set of $n \times n$ idempotent zero patterns over $F$. A matrix with entries from the set $\{+, -, 0\}$ is called a *sign pattern*. The corresponding problem on sign patterns has been extensively studied. See [2] and the references therein. In this paper, we will solve Problem 1 and a related problem.

## 2. Main results

We need several lemmas to establish the main results. If $A$ is a matrix, $A(i, j)$ denotes its entry in the $i$-th row and $j$-th column.

**Lemma 1.** *Let $F$ be a field with at least three elements and let $A \in \Omega_n(F)$. If $A(i, j) = A(j, k) = *$, then $A(i, k) = *$.*

**Proof.** If $j = i$ or $k$, then the result holds trivially.

Next suppose $j \neq i, k$. It suffices to exhibit a matrix $B \in Z_F(A)$ with $B^2(i, k) \neq 0$. Let all the nonzero entries of $B$ except $B(i, j)$ be 1. Since $F$ has at least two nonzero elements, from $B^2(i, k) = B(i, j)B(j, k) + \sum_{l \neq j} B(i, l)B(l, k)$ we deduce that there is a nonzero value of $B(i, j)$ in $F$ such that $B^2(i, k) \neq 0$. Since $A \in \Omega_n(F)$, $B(i, k) \neq 0$. Thus $A(i, k) = *$. □

Recall that in a digraph, a sequence of successively adjacent arcs is called a *walk*. The number of arcs in a walk is called the *length* of the walk. We denote by $D(A)$ the digraph of a matrix $A$ of order $n$. The vertices of $D(A)$ are $1, 2, \ldots, n$ and $(i, j)$ is an arc if and only if $A(i, j) \neq 0$.

**Lemma 2.** *Let $F$ be a field with at least three elements and let $A \in \Omega_n(F)$. If $A(i, j) = 0$, then there is no walk from $i$ to $j$ in $D(A)$.*

**Proof.** To the contrary, assume that there is a walk of length $k$ from $i$ to $j$. Since $A(i, j) = 0$, $(i, j)$ is not an arc in $D(A)$. Then $k \geqslant 2$ and there exist $i_1, \ldots, i_{k-1}$ such that $A(i, i_1) = \cdots = A(i_{k-1}, j) = *$. Applying Lemma 1 $k-1$ times we deduce $A(i, j) = *$, a contradiction. Thus there is no walk from $i$ to $j$ in $D(A)$. □