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Nearly positive matrices

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ABSTRACT

Nearly positive matrices are nonnegative matrices which, when premultiplied by orthogonal matrices as close to the identity as one wishes, become positive. In other words, all columns of a nearly positive matrix are mapped simultaneously into the interior of the nonnegative cone by multiplication by a sequence of orthogonal matrices converging to the identity. In this paper, nearly positive matrices are analyzed and characterized in several cases. Some necessary and some sufficient conditions for a nonnegative matrix to be nearly positive are presented. A connection to completely positive matrices is also presented.

1. Introduction

Consider the cone of nonnegative vectors in the m-dimensional space $\mathbb{R}^m_+ = \{\mathbf{x} \in \mathbb{R}^m, \mathbf{x} \geqslant \mathbf{0}\}$. Any nonzero vector $\mathbf{v}$ in its boundary is nonnegative, but not positive. It is not hard to see that an infinitesimal rotation in the appropriate direction can bring this

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vector into the interior of the cone $\mathbb{R}^m_+$. In other words, one can build a sequence of orthogonal matrices $Q(\ell)$ such that $\lim_{\ell\to\infty} Q(\ell) = I$ with the property that $Q(\ell)\mathbf{v} > \mathbf{0}$ for every ℓ.

For two non-orthogonal nonnegative vectors $\mathbf{u}$ and $\mathbf{v}$, one can also build a sequence of orthogonal matrices, such that *both* $Q(\ell)\mathbf{u} > \mathbf{0}$ and $Q(\ell)\mathbf{v} > \mathbf{0}$ for every ℓ [7, Theorem 6.12]. The existence of such a sequence was used in [7] to study topological properties of the set of matrices having a Perron–Frobenius property; see also [6].

Several natural questions arise from the above-mentioned results. The first of such questions which we address in this paper is: Can one build such a sequence to bring any set of *more than two* non-orthogonal vectors in the boundary of the nonnegative cone into its interior simultaneously? As we shall see, the answer is 'yes' for up to three vectors, but 'no' for four or more vectors. More specifically, let us call an $m \times n$ matrix A *nearly positive* provided there exists a sequence of orthogonal matrices $Q(\ell)$ such that

$$\lim_{\ell\to\infty} Q(\ell) = I \quad \text{and} \quad Q(\ell)A > O \quad \text{for every } \ell,$$

where the last inequality is understood entrywise. Note that the condition $Q(\ell)A > O$ is equivalent to $Q(\ell)\mathbf{a}_i > \mathbf{0}$ for every i, where $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are the columns of A. In this paper we characterize nearly positive matrices, and study their properties. In particular, we present some necessary and some sufficient conditions for a nonnegative matrix to be nearly positive. A connection to completely positive matrices is also presented, and used to deduce certain results on nearly positive matrices.

We use the following notation: O denotes a zero matrix, I the identity matrix, and J a matrix of all ones. When we want to stress the order or size of the matrix we add it as a subscript; e.g. I_n stands for the $n \times n$ identity matrix and $O_{m\times n}$ is the $m \times n$ zero matrix. A vector of all ones is denoted by $\mathbf{1}$, and a zero vector is denoted by $\mathbf{0}$. The Hadamard (entrywise) product of two matrices A and B of the same order is denoted by $A \circ B$, and the direct sum of two matrices by $A \oplus B$. The inner product of two matrices of the same order is the Frobenius inner product $\langle A, B\rangle = \operatorname{trace}(AB^T)$. Whenever we consider a norm of a matrix, we mean the Frobenius norm $\|A\| = \sqrt{\operatorname{trace}(AA^T)}$.

2. A necessary condition

In this section, we present a simple necessary condition, together with some sufficient conditions for a nonnegative matrix to be nearly positive.

We begin with a few simple observations.

Proposition 2.1. *Let A be an $m \times n$ nonnegative matrix.*

(a) *If P is an $m \times m$ permutation matrix, then A is nearly positive if and only if PA is nearly positive.*