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Graphs associated with matrices over finite fields and their endomorphisms

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A B S T R A C T

Let $\mathbb{F}^{m\times n}$ be the set of $m\times n$ matrices over a field $\mathbb{F}$. Consider a graph $G=(\mathbb{F}^{m\times n},\sim)$ with $\mathbb{F}^{m\times n}$ as the vertex set such that two vertices $A,B\in\mathbb{F}^{m\times n}$ are adjacent if $\operatorname{rank}(A-B)=1$. We study graph properties of G when $\mathbb{F}$ is a finite field. In particular, G is a regular connected graph with diameter equal to $\min\{m,n\}$; it is always Hamiltonian. Furthermore, we determine the independence number, chromatic number and clique number of G. These results are used to characterize the graph endomorphisms of G, which extends Hua's fundamental theorem of geometry on $\mathbb{F}^{m\times n}$.

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Independence number
Chromatic number

1. Introduction

Let $\mathbb{F}$ be a field and $\mathbb{F}^{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}$. Define a metric d on $\mathbb{F}^{m\times n}$ by

$$d(A, B) = \operatorname{rank}(A - B).$$

Two matrices $A, B \in \mathbb{F}^{m\times n}$ are *adjacent*, denoted by $A \sim B$, if $d(A, B) = \operatorname{rank}(A - B) = 1$. This metric and adjacency relation give rise to an interesting geometrical structure on $\mathbb{F}^{m\times n}$.

In mid 1940s, Hua initiated the study of the fundamental theorem of the geometry of matrices that concerns the characterization of maps $\phi : \mathbb{F}^{m\times n} \to \mathbb{F}^{m\times n}$ leaving invariant the adjacency relation, i.e., $\operatorname{rank}(\phi(A) - \phi(B)) = 1$ whenever $\operatorname{rank}(A - B) = 1$. Hua also considered the problem on matrices over a division ring, and his study generated considerable interest and led to many interesting results; for example, see [8,9,11,13,18,20].

Suppose $\mathbb{F}$ is the finite field $\mathbb{F}_q$ with q elements. Then the adjacency relation $A \sim B$ in ${\mathbb{F}_q}^{m\times n}$ defined above, i.e., $A \sim B$ if $\operatorname{rank}(A-B) = 1$, gives rise to a graph $G = (V, \sim)$ with $V = {\mathbb{F}_q}^{m\times n}$ as the vertex set and there is an edge joining $A, B \in V$ if and only if $A \sim B$. We call $G = ({\mathbb{F}_q}^{m\times n}, \sim)$ a *matrix graph*, which is also called a *bilinear forms graph* in graph theory. This graph has a lot of interesting properties. For example, it is easy to check that G is a regular graph with diameter equal to $\min\{m, n\}$; it is Eulerian if and only if q is odd. We will give an easy constructive proof to show that G is Hamiltonian. Furthermore, we determine the independence number, chromatic number and clique number of G; see Section 2.

Note that in graph theory literature, it is common to write $G = (V, E)$ with V as the vertex set, and E as the edge set consisting of all the unordered pairs of vertices u and v that are adjacent.

Recall that for two given graphs $G = (V, \sim)$ and $G' = (V', \sim')$, a map $\phi : V \to V'$ is a *graph homomorphism* if

$$\phi(a) \sim' \phi(b) \quad \text{in } G' \quad \text{whenever} \quad a \sim b \text{ in } G.$$

A graph homomorphism is called a *graph endomorphism* if $G = G'$. Thus, the fundamental theorem of geometry of ${\mathbb{F}_q}^{m\times n}$ can be formulated in terms of graph endomorphisms on $({\mathbb{F}_q}^{m\times n}, \sim)$. In [16,17], the author characterized the graph endomorphisms on symmetric matrix graphs and hermitian matrix graphs over a finite field. In Section 3, we will characterize graph endomorphisms on matrix graphs using results in Section 2.