Linear Algebra and its Applications 447 (2014) 26–37

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Further refinements of the Heinz inequality

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ARTICLE INFO

Article history:
Received 3 September 2012
Accepted 8 January 2013
Available online 27 February 2013

Submitted by Hans Schneider

Dedicated in the memory of Professors Michael Neumann and Uriel Rothblum.

AMS classification:
15A60
47A30
47A64
47B15

Keywords:
Heinz inequality
Convex function
Hermite–Hadamard inequality
Positive definite matrix
Unitarily invariant norm

ABSTRACT

The celebrated Heinz inequality asserts that $2|||A^{1/2}XB^{1/2}||| \leqslant |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leqslant |||AX + XB|||$ for $X \in \mathbb{B}(\mathscr{H})$, $A, B \in \mathbb{B}(\mathscr{H})_+$, every unitarily invariant norm $|||\cdot|||$ and $\nu \in [0,1]$. In this paper, we present several improvement of the Heinz inequality by using the convexity of the function $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$, some integration techniques and various refinements of the Hermite–Hadamard inequality. In the setting of matrices we prove that

$$\left|\left|\left|A^{\frac{\alpha+\beta}{2}}XB^{1-\frac{\alpha+\beta}{2}} + A^{1-\frac{\alpha+\beta}{2}}XB^{\frac{\alpha+\beta}{2}}\right|\right|\right|$$
$$\leqslant \frac{1}{|\beta-\alpha|}\left|\left|\left|\int_\alpha^\beta \left(A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right) d\nu\right|\right|\right|$$
$$\leqslant \frac{1}{2}\left|\left|\left|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha + A^\beta XB^{1-\beta} + A^{1-\beta}XB^\beta\right|\right|\right|,$$

for real numbers α, β.

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1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra

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0024-3795/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.laa.2013.01.012

$\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field. The cone of positive operators is denoted by $\mathbb{B}(\mathscr{H})_+$. A unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $\mathfrak{J}_{|||\cdot|||}$ of $\mathbb{B}(\mathscr{H})$ associated with it and has the property $|||UXV||| = |||X|||$, where U and V are unitaries and $X \in \mathfrak{J}_{|||.|||}$. Whenever we write $|||X|||$, we mean that $X \in \mathfrak{J}_{|||.|||}$. The operator norm on $\mathbb{B}(\mathscr{H})$ is denoted by $\|\cdot\|$.

The arithmetic–geometric mean inequality for two positive real numbers a, b is $\sqrt{ab} \leqslant (a+b)/2$, which has been generalized in the context of bounded linear operators as follows. For $A, B \in \mathbb{B}(\mathscr{H})_+$ and an unitarily invariant norm $|||\cdot|||$ it holds that

$$2|||A^{1/2}XB^{1/2}||| \leqslant |||AX + XB|||.$$

For $0 \leqslant \nu \leqslant 1$ and two nonnegative real numbers a and b, the *Heinz mean* is defined as

$$H_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}.$$

The function H_ν is symmetric about the point $\nu = \frac{1}{2}$. Note that $H_0(a, b) = H_1(a, b) = \frac{a+b}{2}$, $H_{1/2}(a, b) = \sqrt{ab}$ and

$$H_{1/2}(a, b) \leqslant H_\nu(a, b) \leqslant H_0(a, b) \tag{1.1}$$

for $0 \leqslant \nu \leqslant 1$, i.e., the Heinz means interpolate between the geometric mean and the arithmetic mean. The generalization of (1.1) in $B(\mathscr{H})$ asserts that for operators A, B, X such that $A, B \in \mathbb{B}(\mathscr{H})_+$, every unitarily invariant norm $|||\cdot|||$ and $\nu \in [0, 1]$ the following double inequality due to Bhatia and Davis [3] holds

$$2|||A^{1/2}XB^{1/2}||| \leqslant |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leqslant |||AX + XB|||. \tag{1.2}$$

Indeed, it has been proved that $F(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$ is a convex function of ν on $[0, 1]$ with symmetry about $\nu = 1/2$, which attains its minimum there at and its maximum at $\nu = 0$ and $\nu = 1$.

The second part of the previous inequality is one of the most essential inequalities in the operator theory, which is called *the Heinz inequality*; see [10]. The proof given by Heinz [11] is based on the complex analysis and is somewhat complicated. In [18], McIntosh showed that the Heinz inequality is a consequence of the following inequality

$$\|A^*AX + XBB^*\| \geqslant 2\,\|AXB\|\ ,$$

where $A, B, X \in \mathbb{B}(\mathscr{H})$. In the literature, the above inequality is called the *arithmetic–geometric mean inequality*. Fujii et al. [9] proved that the Heinz inequality is equivalent to several other norm inequalities such as the *Corach–Porta–Recht inequality* $\|AXA^{-1} + A^{-1}XA\| \geqslant 2\|X\|$, where A is a selfadjoint invertible operator and X is a selfadjoint operator; see also [6]. Audenaert [2] gave a singular value inequality for Heinz means by showing that if $A, B \in \mathcal{M}_n$ are positive semidefinite and $0 \leqslant \nu \leqslant 1$, then $s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \leqslant s_j(A+B)$ for $j = 1, \ldots, n$, where s_j denotes the jth singular value. Also, Yamazaki [22] used the classical Heinz inequality $\|AXB\|^r\|X\|^{1-r} \geq \|A^rXB^r\|$ $(A, B, X \in \mathbb{B}(\mathscr{H}), A \geqslant 0, B \geqslant 0, r \in [0, 1])$ to characterize the chaotic order relation and to study isometric Aluthge transformations.

For a detailed study of these and associated norm inequalities along with their history of origin, refinements and applications, one may refer to [3–5,12–15].

It should be noticed that $F(1/2) \leqslant F(\nu) \leqslant \frac{F(0)+F(1)}{2}$ provides a refinement to the Jensen inequality $F(1/2) \leqslant \frac{F(0)+F(1)}{2}$ for the function F. Therefore it seems quite reasonable to obtain a new refinement of (1.2) by utilizing a refinement of Jensen's inequality. This idea was recently applied by Kittaneh [17] in virtue of the Hermite–Hadamard inequality (2.1).

One of the purposes of the present article is to obtain some new refinements of (1.2), from different refinements of inequality (2.1). We also aim to give a unified study and further refinements to the recent works for matrices.

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