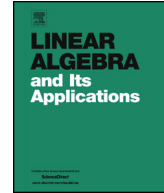




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Generalized inverses of Markovian kernels in terms of properties of the Markov chain



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ABSTRACT

All one-condition generalized inverses of the Markovian kernel $I - P$, where P is the transition matrix of a finite irreducible Markov chain, can be uniquely specified in terms of the stationary probabilities and the mean first passage times of the underlying Markov chain. Special sub-families include the group inverse of $I - P$, Kemeny and Snell's fundamental matrix of the Markov chain and the Moore–Penrose g -inverse. The elements of some sub-families of the generalized inverses can also be re-expressed involving the second moments of the recurrence time variables. Some applications to Kemeny's constant and perturbations of Markov chains are also considered.

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1. Introduction

Let $P = [p_{ij}]$ be the transition matrix of a finite irreducible, discrete time Markov chain $\{X_n\}$ ($n \geq 0$), with state space $S = \{1, 2, \dots, m\}$. Such chains have a unique stationary distribution $\{\pi_j\}$ ($1 \leq j \leq m$). Let $T_{ij} = \min\{n \geq 1, X_n = j \mid X_0 = i\}$ be the first passage time from state i to state j (first return when $i = j$) and define $m_{ij} = E[T_{ij} \mid X_0 = i]$ as the mean first passage time from state i

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to state j (or mean recurrence time of state i when $i = j$). It is well known that for finite irreducible chains all the m_{ij} are well defined and finite.

Generalized matrix inverses (g-inverses) of $I - P$ are typically used to solve systems of linear equations including a variety of the properties of the Markov chain. In particular the $\{\pi_j\}$ and the $\{m_{ij}\}$ can be found in terms of g-inverses, either in matrix form or in terms of the elements of the g-inverse. What is not known is that the elements of every g-inverse of $I - P$ can be expressed in terms of the stationary probabilities $\{\pi_j\}$ and the mean first passage times $\{m_{ij}\}$ of the associated Markov chain. The key thrust to this paper is to first identify the parameters that characterize different sub-families of g-inverses of $I - P$. Then to assign to each sub-family, thus characterized, explicit expressions for the elements of the g-inverses in terms of the $\{\pi_j\}$ and the $\{m_{ij}\}$.

2. Generalized inverses of the Markovian kernel $I - P$

A g-inverse of a matrix A is any matrix A^- such that $AA^-A = A$. Matrices A^- are called “one-condition” g-inverses or “equation solving” g-inverses because of their use in solving systems of linear equations.

If A is non-singular then A^- is A^{-1} , the inverse of A , and is unique. If A is singular, A^- is not unique. Typically, in the equations that we wish to solve, we only need a one-condition g-inverse with the non-uniqueness being eliminated by the imposition of boundary conditions (such as $\sum_{i=1}^m \pi_i = 1$ in the case of finding stationary distributions, and $m_{ii} = 1/\pi_i$ in the case of mean first passage times).

The following theorem in [5] gives a procedure for finding all one-condition g-inverses of $I - P$.

Theorem 1. *Let P be the transition matrix of a finite irreducible Markov chain with m states and stationary probability vector $\pi^T = (\pi_1, \pi_2, \dots, \pi_m)$.*

Let $e^T = (1, 1, \dots, 1)$ and t and u be any vectors.

- (a) $I - P + tu^T$ is non-singular if and only if $\pi^T t \neq 0$ and $u^T e \neq 0$.
- (b) If $\pi^T t \neq 0$ and $u^T e \neq 0$ then $[I - P + tu^T]^{-1}$ is a one-condition g-inverse of $I - P$ and, further, all “one-condition” g-inverses of $I - P$ can be expressed as $A^{(1)} = [I - P + tu^T]^{-1} + e f^T + g \pi^T$ for arbitrary vectors f and g .

Useful by-products of the proof of the above theorem were the following results:

$$[I - P + tu^T]^{-1} t = \frac{e}{u^T e} \tag{2.1}$$

$$u^T [I - P + tu^T]^{-1} = \frac{\pi^T}{\pi^T t} \tag{2.2}$$

Special multi-condition g-inverses of A can also be considered by imposing additional conditions. Consider real conformable matrices X (which in our context we assume to be square) such that:

- (Condition 1) $AXA = A$.
- (Condition 2) $XAX = X$.
- (Condition 3) $(AX)^T = AX$.
- (Condition 4) $(XA)^T = XA$.
- (Condition 5) $AX = XA$.

Let $A^{(i,j,\dots,l)}$ be any matrix that satisfies conditions (i), (j), ..., (l) of the above itemised conditions. $A^{(i,j,\dots,l)}$ is called an (i, j, \dots, l) g-inverse of A , under the assumption that condition 1 is always included. Let $A\{i, j, \dots, l\}$ be the class of all (i, j, \dots, l) g-inverses of A .

The classification of g-inverses of the Markovian kernel $I - P$, can be done simply by means of the following results given in [8].

Theorem 2. *If G is any g-inverse of $I - P$, where P is the transition matrix of a finite irreducible Markov chain with stationary probability vector π^T , then G can be uniquely expressed in parametric form as*

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