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Generalized inverses of Markovian kernels in terms of properties of the Markov chain

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ABSTRACT

All one-condition generalized inverses of the Markovian kernel $I - P$, where P is the transition matrix of a finite irreducible Markov chain, can be uniquely specified in terms of the stationary probabilities and the mean first passage times of the underlying Markov chain. Special sub-families include the group inverse of $I - P$, Kemeny and Snell's fundamental matrix of the Markov chain and the Moore–Penrose g-inverse. The elements of some sub-families of the generalized inverses can also be re-expressed involving the second moments of the recurrence time variables. Some applications to Kemeny's constant and perturbations of Markov chains are also considered.

1. Introduction

Let $P = [p_{ij}]$ be the transition matrix of a finite irreducible, discrete time Markov chain $\{X_n\}$ $(n \geqslant 0)$, with state space $S = \{1, 2, \ldots, m\}$. Such chains have a unique stationary distribution $\{\pi_j\}$ $(1 \leqslant j \leqslant m)$. Let $T_{ij} = \min[n \geqslant 1,\ X_n = j \mid X_0 = i]$ be the first passage time from state i to state j (first return when $i = j$) and define $m_{ij} = E[T_{ij} \mid X_0 = i]$ as the mean first passage time from state i

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to state j (or mean recurrence time of state i when $i = j$). It is well known that for finite irreducible chains all the m_{ij} are well defined and finite.

Generalized matrix inverses (g-inverses) of $I - P$ are typically used to solve systems of linear equations including a variety of the properties of the Markov chain. In particular the $\{\pi_j\}$ and the $\{m_{ij}\}$ can be found in terms of g-inverses, either in matrix form or in terms of the elements of the g-inverse. What is not known is that the elements of every g-inverse of $I - P$ can be expressed in terms of the stationary probabilities $\{\pi_j\}$ and the mean first passage times $\{m_{ij}\}$ of the associated Markov chain. The key thrust to this paper is to first identify the parameters that characterize different sub-families of g-inverses of $I - P$. Then to assign to each sub-family, thus characterized, explicit expressions for the elements of the g-inverses in terms of the $\{\pi_j\}$ and the $\{m_{ij}\}$.

2. Generalized inverses of the Markovian kernel $I - P$

A g-inverse of a matrix A is any matrix A^- such that $AA^-A = A$. Matrices A^- are called "one-condition" g-inverses or "equation solving" g-inverses because of their use in solving systems of linear equations.

If A is non-singular then A^- is A^{-1}, the inverse of A, and is unique. If A is singular, A^- is not unique. Typically, in the equations that we wish to solve, we only need a one-condition g-inverse with the non-uniqueness being eliminated by the imposition of boundary conditions (such as $\sum_{i=1}^{m} \pi_i = 1$ in the case of finding stationary distributions, and $m_{ii} = 1/\pi_i$ in the case of mean first passage times).

The following theorem in [5] gives a procedure for finding all one-condition g-inverses of $I - P$.

Theorem 1. *Let P be the transition matrix of a finite irreducible Markov chain with m states and stationary probability vector $\boldsymbol{\pi}^T = (\pi_1, \pi_2, \ldots, \pi_m)$.*

Let $\boldsymbol{e}^T = (1, 1, \ldots, 1)$ and $\boldsymbol{t}$ and $\boldsymbol{u}$ be any vectors.

(a) *$I - P + \boldsymbol{t}\boldsymbol{u}^T$ is non-singular if and only if $\boldsymbol{\pi}^T\boldsymbol{t} \neq 0$ and $\boldsymbol{u}^T\boldsymbol{e} \neq 0$.*
(b) *If $\boldsymbol{\pi}^T\boldsymbol{t} \neq 0$ and $\boldsymbol{u}^T\boldsymbol{e} \neq 0$ then $[I - P + \boldsymbol{t}\boldsymbol{u}^T]^{-1}$ is a one-condition g-inverse of $I - P$ and, further, all "one-condition" g-inverses of $I - P$ can be expressed as $A^{(1)} = [I - P + \boldsymbol{t}\boldsymbol{u}^T]^{-1} + \boldsymbol{e}\boldsymbol{f}^T + \boldsymbol{g}\boldsymbol{\pi}^T$ for arbitrary vectors $\boldsymbol{f}$ and $\boldsymbol{g}$.*

Useful by-products of the proof of the above theorem were the following results:

$$\left[I - P + \boldsymbol{t}\boldsymbol{u}^T\right]^{-1}\boldsymbol{t} = \frac{\boldsymbol{e}}{\boldsymbol{u}^T\boldsymbol{e}}. \tag{2.1}$$

$$\boldsymbol{u}^T\left[I - P + \boldsymbol{t}\boldsymbol{u}^T\right]^{-1} = \frac{\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T\boldsymbol{t}}. \tag{2.2}$$

Special multi-condition g-inverses of A can also be considered by imposing additional conditions. Consider real conformable matrices X (which in our context we assume to be square) such that:

(Condition 1) $AXA = A$.
(Condition 2) $XAX = X$.
(Condition 3) $(AX)^T = AX$.
(Condition 4) $(XA)^T = XA$.
(Condition 5) $AX = XA$.

Let $A^{(i,j,\ldots,l)}$ be any matrix that satisfies conditions $(i), (j), \ldots, (l)$ of the above itemised conditions. $A^{(i,j,\ldots,l)}$ is called an $(i, j, \ldots, l)$ g-inverse of A, under the assumption that condition 1 is always included. Let $A\{i, j, \ldots, l\}$ be the class of all $(i, j, \ldots, l)$ g-inverses of A.

The classification of g-inverses of the Markovian kernel $I - P$, can be done simply by means of the following results given in [8].

Theorem 2. *If G is any g-inverse of $I - P$, where P is the transition matrix of a finite irreducible Markov chain with stationary probability vector $\boldsymbol{\pi}^T$, then G can be uniquely expressed in parametric form as*