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Convex integer optimization by constantly many linear counterparts

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A R T I C L E I N F O

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Dedicated to the memory of Uri Rothblum

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A B S T R A C T

In this article we study convex integer maximization problems with composite objective functions of the form $f(Wx)$, where f is a convex function on $\mathbb{R}^d$ and W is a $d \times n$ matrix with small or binary entries, over finite sets $S \subset \mathbb{Z}^n$ of integer points presented by an oracle or by linear inequalities.

Continuing the line of research advanced by Uri Rothblum and his colleagues on edge-directions, we introduce here the notion of *edge complexity* of S, and use it to establish polynomial and constant upper bounds on the number of vertices of the projection $\mathrm{conv}(WS)$ and on the number of linear optimization counterparts needed to solve the above convex problem.

Two typical consequences are the following. First, for any d, there is a constant $m(d)$ such that the maximum number of vertices of the projection of any matroid $S \subset \{0,1\}^n$ by any binary $d \times n$ matrix W is $m(d)$ regardless of n and S; and the convex matroid problem reduces to $m(d)$ greedily solvable linear counterparts. In particular, $m(2) = 8$. Second, for any d, l, m, there is a constant $t(d; l, m)$ such that the maximum number of vertices of the projection of any three-index $l \times m \times n$ transportation polytope for any n by any binary $d \times (l \times m \times n)$ matrix W is $t(d; l, m)$; and the convex three-index transportation problem reduces to $t(d; l, m)$ linear counterparts solvable in polynomial time.

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1. Introduction

In this article we study convex integer maximization problems and the closely related projections of the sets of feasible points. Let $S \subset \mathbb{Z}^n$ be a finite set of integer points, let $\text{conv}(S) \subset \mathbb{R}^n$ be its convex hull, let W be a $d \times n$ integer matrix, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. We study the problem of maximizing the composite function $f(Wx)$ over S and the projection of $\text{conv}(S)$ by W into $\mathbb{R}^d$, namely,

$$\max\bigl\{f(Wx)\colon x \in S\bigr\} \tag{1}$$

and

$$\text{conv}(WS) = \text{conv}\{Wx\colon x \in S\} = \bigl\{Wx\colon x \in \text{conv}(S)\bigr\} \subset \mathbb{R}^d. \tag{2}$$

The sets S we consider arise in two natural contexts. First, in *combinatorial optimization*, in which case $S \subseteq \{0,1\}^n$ has some combinatorial structure and might be presented by a suitable oracle. Second, in *integer programming*, where

$$S := \bigl\{x \in \mathbb{Z}^n,\ Ax = b,\ l \leqslant x \leqslant u\bigr\} \tag{3}$$

is the set of integer points satisfying a given (standard) system of linear inequalities.

The optimization problem (1) can also be interpreted as a problem of multicriteria optimization, where each row of W gives a linear criterion $W_i x$ and f compromises these criteria. We therefore call W the *criteria* matrix or *weight* matrix.

The projection polytope $\text{conv}(WS)$ in (2) and its vertices play a central role in solving problem (1): for any convex function f there is an optimal solution x whose projection $y := Wx$ is a vertex of $\text{conv}(WS)$. In particular, the enumeration of all vertices of $\text{conv}(WS)$ enables to compute the optimal objective value for any given convex function f by picking that vertex attaining the best value $f(y) = f(Wx)$. So it suffices, and will be assumed throughout, that f is presented by a *comparison oracle* that, queried on vectors $y, z \in \mathbb{R}^d$, asserts whether or not $f(y) < f(z)$.

This line of research was advanced by Uri Rothblum, to whom we dedicate this article, and his colleagues, in several papers including [2,5,13], and culminated in the edge-direction framework of [16], see also [15, Chapter 2]. In this article we continue this line of investigation, and take a closer look on *coarse* criteria matrices; that is, we assume that the entries of W are small, presented in unary, or even bounded by a constant and lie in $\{0, 1, \ldots, p\}$. In multicriteria combinatorial optimization, this corresponds to the weight $W_{i,j}$ attributed to element j of the ground set $\{1, \ldots, n\}$ under criterion i being a small or even $\{0, 1\}$ value for all i, j.

Here is a typical result we obtain in convex combinatorial optimization, where $S \subset \{0,1\}^n$ is the set of indicating vectors of bases of a matroid over $\{1, \ldots, n\}$.