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Structured mapping problems for linearly structured matrices

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ABSTRACT

Given an appropriate class of structured matrices $\mathbb{S}$, we characterize matrices X and B for which there exists a matrix $A \in \mathbb{S}$ such that $AX = B$ and determine all matrices in $\mathbb{S}$ mapping X to B. We also determine all matrices in $\mathbb{S}$ mapping X to B and having the smallest norm. We use these results to investigate structured backward errors of approximate eigenpairs and approximate invariant subspaces, and structured pseudospectra of structured matrices.

1. Introduction

Consider a stable linear time-invariant (LTI) control system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = 0, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

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with $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times p}$, $C \in \mathbb{K}^{p\times n}$ and $D \in \mathbb{K}^{p\times p}$. Here $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, u is the input, x is the state and y is the output. The system (1) is said to be passive if the *Hamiltonian matrix*

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} := \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^* \\ -C^*R^{-1}C & -(A - BR^{-1}C)^* \end{bmatrix} \tag{2}$$

has no purely imaginary eigenvalues, where $R := D + D^*$, see [3,6,2]. A matrix $\mathcal{H} \in \mathbb{K}^{2n\times 2n}$ of the form $\mathcal{H} = \left[\begin{smallmatrix} A & F \\ G & -A^* \end{smallmatrix}\right]$ is called Hamiltonian, where $G^* = G$ and $F^* = F$. Equivalently, $\mathcal{H}$ is Hamiltonian $\Longleftrightarrow (\mathcal{J}\mathcal{H})^* = \mathcal{J}\mathcal{H}$, where $\mathcal{J} := \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ and I the identity matrix of size n.

For passivation problem, when purely imaginary eigenvalues occur, one tries to perturb $\mathcal{H}$ by a Hamiltonian matrix $\mathcal{E}$ with *small norm* so that the perturbed matrix $\mathcal{H} + \mathcal{E}$ has no purely imaginary eigenvalues. If such an $\mathcal{E}$ exists, then for some $X \in \mathbb{K}^{2n\times p}$ and $D \in \mathbb{K}^{p\times p}$, we have

$$(\mathcal{H} + \mathcal{E})X = XD \quad \Longrightarrow \quad \mathcal{E}X = B := \mathcal{H}X - XD.$$

This leads us to the following mapping problem.

Problem 1 (*Hamiltonian mapping problem*). Given $X, B \in \mathbb{K}^{2n\times p}$, consider

$$\mathrm{Ham}(X, B) := \left\{\mathcal{H} \in \mathbb{K}^{2n\times 2n}\colon (\mathcal{J}\mathcal{H})^* = \mathcal{J}\mathcal{H} \text{ and } \mathcal{H}X = B\right\},$$
$$\sigma^{\mathrm{Ham}}(X, B) := \inf\left\{\|\mathcal{H}\|\colon \mathcal{H} \in \mathrm{Ham}(X, B)\right\}.$$

- Characterize $X, B \in \mathbb{K}^{2n\times p}$ for which $\mathrm{Ham}(X, B) \neq \emptyset$ and determine all matrices in $\mathrm{Ham}(X, B)$.
- Also determine all optimal solutions $\mathcal{H}_o \in \mathrm{Ham}(X, B)$ such that $\|\mathcal{H}_o\| = \sigma^{\mathrm{Ham}}(X, B)$.

Motivated by Problem 1, we now consider structured mapping problem for various classes of structured matrices. Let $\mathbb{S}$ denote a class of structured matrices in $\mathbb{K}^{n\times n}$. The class $\mathbb{S}$ we consider in this paper is either a Jordan or a Lie algebra associated with an appropriate scalar product on $\mathbb{K}^n$. This provides a general setting that encompasses important classes of structured matrices such as Hamiltonian, skew-Hamiltonian, symmetric, skew-symmetric, pseudosymmetric, persymmetric, Hermitian, skew-Hermitian, pseudo-Hermitian, pseudo-skew-Hermitian, to name only a few, see [9]. We, therefore, consider the following problem.

Problem 2 (*Structured Mapping Problem*). Let $\mathbb{S} \subset \mathbb{K}^{n\times n}$ be a class of structured matrices and let $X, B \in \mathbb{K}^{n\times p}$. Set

$$\mathbb{S}(X, B) := \{A \in \mathbb{S}\colon AX = B\},$$
$$\sigma^{\mathbb{S}}(X, B) := \inf\left\{\|A\|\colon A \in \mathbb{S}(X, B)\right\}.$$

- **Existence:** Characterize $X, B \in \mathbb{K}^{n\times p}$ for which $\mathbb{S}(X, B) \neq \emptyset$.
- **Characterization:** Determine all matrices in $\mathbb{S}(X, B)$. Also determine all optimal solutions $A_o \in \mathbb{S}(X, B)$ such that $\|A_o\| = \sigma^{\mathbb{S}}(X, B)$.

We mention that structured backward error of an approximate invariant subspace of a structured matrix also leads to a structured mapping problem. A subspace $\mathcal{X}$ is invariant under A if $A\mathcal{X} \subset \mathcal{X}$.

Problem 3 (*Structured backward error*). Let $\mathbb{S} \subset \mathbb{K}^{n\times n}$ be a class of structured matrices and $A \in \mathbb{S}$. Let $\mathcal{X}$ be a subspace of $\mathbb{K}^n$. Set

$$\omega^{\mathbb{S}}(A, \mathcal{X}) := \min\left\{\|\Delta A\|\colon \Delta A \in \mathbb{S} \text{ and } (A + \Delta A)\mathcal{X} \subset \mathcal{X}\right\}.$$

Find all $E \in \mathbb{S}$ such that $(A + E)\mathcal{X} \subset \mathcal{X}$ and $\|E\| = \omega^{\mathbb{S}}(A, \mathcal{X})$.