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Eigenvalue multiplicity in cubic graphs

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ABSTRACT

Let G be a connected cubic graph of order n with μ as an eigenvalue of multiplicity k. We show that (i) if $\mu \notin \{-1, 0\}$ then $k \leqslant \frac{1}{2}n$, with equality if and only if $\mu = 1$ and G is the Petersen graph; (ii) if $\mu = -1$ then $k \leqslant \frac{1}{2}n + 1$, with equality if and only if $G = K_4$; (iii) if $\mu = 0$ then $k \leqslant \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.

1. Introduction

Let G be a regular graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k and codimension t. From [1, Theorem 3.1], we know that if $\mu \notin \{-1, 0\}$ and $t > 2$ then $k \leqslant n - \frac{1}{2}(-1 + \sqrt{8n+9})$, equivalently $k \leqslant \frac{1}{2}(t+1)(t-2)$. For cubic graphs, this quadratic bound improves an earlier cubic bound noted in [4, p. 162]. In fact, when $\mu \neq 0$ and G is connected, a linear bound follows easily from the equation $\mathrm{tr}(A) = 0$. To see this, note first that if $k \geqslant \frac{1}{2}n$ then μ is an integer, for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity $\frac{1}{2}n$. It follows that if G is a connected cubic graph then $\mu \in \{-2, -1, 0, 1, 2\}$ (see [3, Sections 1.3 and 3.2]). If $k = n - 1$ then G is complete, $n = 4$ and $\mu = -1$; otherwise let d be the mean of the eigenvalues other than 3 and μ, so that $3 + k\mu + (n - k - 1)d = 0$. We have $-3 \leqslant d < 3$; moreover, if $d = -3$ then G is bipartite, $k = n - 2$ and $\mu = 0$ (see [3, Theorems 3.2.3 and 3.2.4]). We deduce:

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(a) if $\mu = -2$ then $k < \frac{3}{5}n$, i.e. $k < \frac{3}{2}t$;
(b) if $\mu = -1$ then $k \leqslant \frac{3}{4}n$, i.e. $k \leqslant 3t$;
(c) if $\mu = 0$ then $k \leqslant n - 2$;
(d) if $\mu = 1$ then $k < \frac{3}{4}n - \frac{3}{2}$, i.e. $k < 3t - 6$;
(e) if $\mu = 2$ then $k < \frac{3}{5}n - \frac{6}{5}$, i.e. $k < \frac{3}{2}t - 3$.

We use star complements to improve these bounds, and to determine all the graphs for which the new bounds are attained. Our main result is the following; here and throughout we use the notation of the monograph [3].

Theorem 1.1. *Let G be a connected cubic graph of order n with μ as an eigenvalue of multiplicity k.*

(i) *If $\mu \notin \{-1, 0\}$ then $k \leqslant \frac{1}{2}n$, with equality if and only if $\mu = 1$ and G is the Petersen graph.*
(ii) *If $\mu = -1$ then $k \leqslant \frac{1}{2}n + 1$, with equality if and only if $G = K_4$.*
(iii) *If $\mu = 0$ then $k \leqslant \frac{1}{2}n + 1$, with equality if and only if $G = \overline{2K_3}$.*

It follows that if G is a connected cubic graph of order $n > 10$ with μ as an eigenvalue of multiplicity k then $k \leqslant \frac{1}{2}n - 1$ when $\mu \notin \{-1, 0\}$, and $k \leqslant \frac{1}{2}n$ otherwise.

2. Preliminaries

Let G be a graph of order n with μ as an eigenvalue of multiplicity k. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [3, Chapter 5]. We shall require the following results, where for any $X \subseteq V(G)$, we write G_X for the subgraph of G induced by X. We take $V(G) = \{1, \dots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent.

Theorem 2.1. *(See [3, Theorem 5.1.7].) Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*

(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top (\mu I - C)^{-1} B. \tag{1}$$

(ii) *If X is a star set for μ then $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1} B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in \mathbb{R}^k)$.*

Let $H = G - X$, where X is a star set for μ. The columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H)\colon\ u \sim v\}$ $(u \in X)$. Eq. (1) shows that

$$\mathbf{b}_u^\top (\mu I - C)^{-1} \mathbf{b}_v = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise,} \end{cases}$$

and we deduce from Theorem 2.1:

Lemma 2.2. *If X is a star set for μ, and $\mu \notin \{-1, 0\}$, then the neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.*

Let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. Since P is a polynomial in A [3, Eq. 1.5] we have $\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i$ $(i = 1, \dots, n)$, whence:

Lemma 2.3. $\mu P\mathbf{e}_i = \sum_{j \sim i} P\mathbf{e}_j$ $(i = 1, \dots, n)$.