Linear Algebra and its Applications 444 (2014) 231–245

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Sharp bounds of the inverse matrices resulted from five-point stencil in solving Poisson equations

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ARTICLE INFO

Article history:
Received 27 October 2013
Accepted 19 November 2013
Available online 7 December 2013
Submitted by R. Brualdi

MSC:
15A09
15A39
35J05
39A12

Keywords:
Poisson equations
Five-point stencil
Matrix inverse
Infinity norm

ABSTRACT

In this paper, we derive the least upper bound (in the infinity norm) and the greatest lower bound of a class of the inverse matrices resulted from the five-point stencil in solving the Poisson equations on the unit square. The obtained bounds are sharp and provide more accurate convergence estimation than the current one in literature. Our approach is based on a matrix theoretic setting which can capture the characteristics of this type of matrices. As an application, we apply the result to the unbiased random walk in a unit square with an absorbing boundary and give the least upper bound of the mean first passage time for an inside particle to reach the boundary.

1. Introduction

In this paper we consider the following matrix

$$-\Delta_h = n^2 \begin{bmatrix} T & -I & 0 & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{bmatrix}_{(n-1)^2\times(n-1)^2} \tag{1.1}$$

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.11.029

where I is the $(n-1)\times(n-1)$ identity matrix and T is an $(n-1)\times(n-1)$ matrix given by

$$T=\begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{bmatrix}_{(n-1)\times(n-1)}, \tag{1.2}$$

where $n \geqslant 2$ is an integer, and the notation $-\Delta_h$ follows from the convention.

The matrix $-\Delta_h$ defined in (1.1) plays an important role in solving the two-dimensional Poisson equation by using finite difference approximation. More specifically, a two-dimensional Poisson equation has the form:

$$-\Delta u = f \quad \text{on } \Omega,$$

along with some given boundary condition. Here Ω is an open bounded domain of $\mathbb{R}^2$ and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian in two dimensions.

Let Ω be an open unit square and $\Omega = \{(x,y),\ 0<x,y<1\}$. We denote the boundary of Ω by $\partial\Omega$. Consider the Poisson equation

$$-\Delta u = f \quad \text{inside } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We take a uniform grid of size $h = 1/n$, where $n \geqslant 2$ is a positive integer. Let

$$\Omega_h = \{x_{ij} = (ih, jh),\ 1 \leqslant i, j \leqslant n-1\}$$

denote the set of interior grid points and

$$\partial\Omega_h = \{(0, jh), (1, jh), (jh, 0), (jh, 1),\ 1 \leqslant j \leqslant n-1\}$$

denote the boundary grid points. The finite difference method seeks the solution of the PDE at the grid points in Ω. Under the above setting, the $(n-1)^2$ unknowns are $u_{ij} = u(ih, jh)$, $1 \leqslant i, j \leqslant n-1$. By applying the Taylor expansions at grid points, one can obtain $(n-1)^2$ equations by approximating the differential equation by a finite difference approximation at each interior point as

$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2} = f_{ij} := f(x_{ij}), \quad 1 \leqslant i, j \leqslant n-1,$$

which is called the five-point stencil in standard numerical analysis textbooks (e.g., see [9]). According to the boundary condition, we know $u_{0j} = u_{nj} = u_{i0} = u_{in} = 0$. If we arrange the unknowns as

$$\mathbf{u}_h = [u_{11}, \dots, u_{n-1,1}, u_{12}, \dots, u_{n-1,2}, \dots, u_{n-1,n-1}]^T$$

and arrange the value of the function f on the grid points as

$$\mathbf{f}_h = [f_{11}, \dots, f_{n-1,1}, f_{12}, \dots, f_{n-1,2}, \dots, f_{n-1,n-1}]^T,$$

then we have the following discrete Poisson equation

$$-\Delta_h \mathbf{u}_h = \mathbf{f}_h$$

which is a system of linear equations, where $-\Delta_h$ is defined by (1.1). It is well known that the matrix $-\Delta_h$ is invertible (we also will show it under our framework). Thus the discrete Poisson equation has a unique solution

$$\mathbf{u}_h = (-\Delta_h)^{-1}\mathbf{f}_h.$$