Linear Algebra and its Applications 444 (2014) 246–253

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the eigenvalues of certain Cayley graphs and arrangement graphs

Bai Fan Chen [a], Ebrahim Ghorbani [b,c], Kok Bin Wong [a,*]

[a] Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
[b] Department of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran
[c] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

ARTICLE INFO

Article history:
Received 30 October 2013
Accepted 25 November 2013
Available online 11 December 2013
Submitted by R. Brualdi

MSC:
05C50
20C10
05A05

Keywords:
Arrangement graph
Cayley graph
Symmetric group
Spectrum integrality

ABSTRACT

In this paper, we show that the eigenvalues of certain classes of Cayley graphs are integers. The (n, k, r)-arrangement graph $A(n, k, r)$ is a graph with all the k-permutations of an n-element set as vertices where two k-permutations are adjacent if they differ in exactly r positions. We establish a relation between the eigenvalues of the arrangement graphs and the eigenvalues of certain Cayley graphs. As a result, the conjecture on integrality of eigenvalues of $A(n, k, 1)$ follows.

1. Introduction

Let Γ be a simple graph with vertex set $\mathcal{V}$. The *adjacency matrix* of Γ is a $\mathcal{V} \times \mathcal{V}$ matrix where its rows and columns indexed by the vertex set of Γ and its (u, v)-entry is 1 if the vertices u and v are adjacent and 0 otherwise. By *eigenvalues* of Γ we mean the eigenvalues of its adjacency matrix. A graph is said to be *integral* if all its eigenvalues are integers. All graphs considered are finite (multi-)graphs without self-loops.

* Corresponding author.
E-mail addresses: tufofo1120@gmail.com (B.F. Chen), e_ghorbani@ipm.ir (E. Ghorbani), kbwong@um.edu.my (K.B. Wong).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.11.035

1.1. Cayley graphs

Let G be a finite group and S be an inverse closed subset of G. The *Cayley graph* $\Gamma(G, S)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = su$ for some $s \in S$.

A Cayley graph $\Gamma(G, S)$ is said to be *normal* if S is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of G.

Theorem 1.1. *(See [2,12,18,19].) The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s\in S} \chi(s),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Let $\mathcal{S}_n$ be the symmetric group on $[n] = \{1, \dots, n\}$ and $S \subseteq \mathcal{S}_n$ be closed under conjugation. Since central characters are algebraic integers [16, Theorem 3.7 on p. 36] and that the characters of the symmetric group are integers ([16, 2.12 on p. 31] or [21, Corollary 2 on p. 103]), by Theorem 1.1, the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ are integers.

Corollary 1.2. *The eigenvalues of a normal Cayley graph $\Gamma(\mathcal{S}_n, S)$ are integers.*

In general, if S is not closed under conjugation, then the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ may not be integers [13] (see also [1,17,20] for related results on the eigenvalues of certain Cayley graphs).

Problem 1.3. Find conditions on S, so that the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ are integers.

Let $2 \leqslant r \leqslant n$ and $\mathrm{Cy}(r)$ be the set of all r cycles in $\mathcal{S}_n$ which do not fix 1, i.e.

$$\mathrm{Cy}(r) = \{\alpha \in \mathcal{S}_n \mid \alpha(1) \neq 1 \text{ and } \alpha \text{ is an } r\text{-cycle}\}.$$

For instance, $\mathrm{Cy}(2) = \{(1\ 2), (1\ 3), \dots, (1\ n)\}$. It was conjectured by Abdollahi and Vatandoost [1] that the eigenvalues of $\Gamma(\mathcal{S}_n, \mathrm{Cy}(2))$ are integers, and contain all integers in the range from $-(n-1)$ to $n-1$ (with the sole exception that when $n = 2$ or 3, zero is not an eigenvalue of $\Gamma(\mathcal{S}_n, \mathrm{Cy}(2))$). The second part of the conjecture was proved by Krakovski and Mohar [17]. In fact, they showed that for $n \geqslant 2$ and each integer $1 \leqslant l \leqslant n-1$, $\pm(n-l)$ are eigenvalues of $\Gamma(\mathcal{S}_n, \mathrm{Cy}(2))$ with multiplicity at least $\binom{n-2}{l-1}$. Furthermore, if $n \geqslant 4$, then 0 is an eigenvalue of $\Gamma(\mathcal{S}_n, \mathrm{Cy}(2))$ with multiplicity at least $\binom{n-1}{2}$. Later, Chapuy and Féray [4] pointed out that the conjecture could be proved by using Jucys–Murphy elements. In this paper, we generalize this to the following:

Theorem 1.4. *The eigenvalues of $\Gamma(\mathcal{S}_n, \mathrm{Cy}(r))$ are integers.*

In fact, Theorem 1.4 follows from Theorem 3.4 which states that for certain subsets S of $\mathcal{S}_n$, the eigenvalues of $\Gamma(\mathcal{S}_n, S)$ are integers.

1.2. Arrangement graphs

For $k \leqslant n$, a *k-permutation* of $[n]$ is an injective function from $[k]$ to $[n]$. So any k-permutation π can be represented by a vector $(i_1, \dots, i_k)$ where $\pi(j) = i_j$ for $j = 1, \dots, k$. Let $1 \leqslant r \leqslant k \leqslant n$. The *$(n, k, r)$-arrangement graph* $A(n, k, r)$ has all the k-permutations of $[n]$ as vertices and two k-permutations are adjacent if they differ in exactly r positions. Formally, the vertex set $V(n, k)$ and edge set $E(n, k, r)$ of $A(n, k, r)$ are