

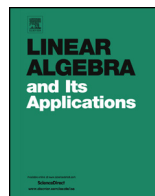


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On the eigenvalues of certain Cayley graphs and arrangement graphs



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ABSTRACT

In this paper, we show that the eigenvalues of certain classes of Cayley graphs are integers. The (n, k, r) -arrangement graph $A(n, k, r)$ is a graph with all the k -permutations of an n -element set as vertices where two k -permutations are adjacent if they differ in exactly r positions. We establish a relation between the eigenvalues of the arrangement graphs and the eigenvalues of certain Cayley graphs. As a result, the conjecture on integrality of eigenvalues of $A(n, k, 1)$ follows.

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1. Introduction

Let Γ be a simple graph with vertex set v . The *adjacency matrix* of Γ is a $v \times v$ matrix where its rows and columns indexed by the vertex set of Γ and its (u, v) -entry is 1 if the vertices u and v are adjacent and 0 otherwise. By *eigenvalues* of Γ we mean the eigenvalues of its adjacency matrix. A graph is said to be *integral* if all its eigenvalues are integers. All graphs considered are finite (multi-)graphs without self-loops.

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1.1. Cayley graphs

Let G be a finite group and S be an inverse closed subset of G . The Cayley graph $\Gamma(G, S)$ is the graph which has the elements of G as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = su$ for some $s \in S$.

A Cayley graph $\Gamma(G, S)$ is said to be *normal* if S is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of G .

Theorem 1.1. (See [2,12,18,19].) *The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are given by*

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where χ ranges over all the irreducible characters of G . Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Let S_n be the symmetric group on $[n] = \{1, \dots, n\}$ and $S \subseteq S_n$ be closed under conjugation. Since central characters are algebraic integers [16, Theorem 3.7 on p. 36] and that the characters of the symmetric group are integers ([16, 2.12 on p. 31] or [21, Corollary 2 on p. 103]), by Theorem 1.1, the eigenvalues of $\Gamma(S_n, S)$ are integers.

Corollary 1.2. *The eigenvalues of a normal Cayley graph $\Gamma(S_n, S)$ are integers.*

In general, if S is not closed under conjugation, then the eigenvalues of $\Gamma(S_n, S)$ may not be integers [13] (see also [1,17,20] for related results on the eigenvalues of certain Cayley graphs).

Problem 1.3. Find conditions on S , so that the eigenvalues of $\Gamma(S_n, S)$ are integers.

Let $2 \leq r \leq n$ and $\text{Cy}(r)$ be the set of all r cycles in S_n which do not fix 1, i.e.

$$\text{Cy}(r) = \{ \alpha \in S_n \mid \alpha(1) \neq 1 \text{ and } \alpha \text{ is an } r\text{-cycle} \}.$$

For instance, $\text{Cy}(2) = \{(1\ 2), (1\ 3), \dots, (1\ n)\}$. It was conjectured by Abdollahi and Vatandoost [1] that the eigenvalues of $\Gamma(S_n, \text{Cy}(2))$ are integers, and contain all integers in the range from $-(n-1)$ to $n-1$ (with the sole exception that when $n=2$ or 3 , zero is not an eigenvalue of $\Gamma(S_n, \text{Cy}(2))$). The second part of the conjecture was proved by Krakovski and Mohar [17]. In fact, they showed that for $n \geq 2$ and each integer $1 \leq l \leq n-1$, $\pm(n-l)$ are eigenvalues of $\Gamma(S_n, \text{Cy}(2))$ with multiplicity at least $\binom{n-2}{l-1}$. Furthermore, if $n \geq 4$, then 0 is an eigenvalue of $\Gamma(S_n, \text{Cy}(2))$ with multiplicity at least $\binom{n-1}{2}$. Later, Chapuy and Féray [4] pointed out that the conjecture could be proved by using Jucys–Murphy elements. In this paper, we generalize this to the following:

Theorem 1.4. *The eigenvalues of $\Gamma(S_n, \text{Cy}(r))$ are integers.*

In fact, Theorem 1.4 follows from Theorem 3.4 which states that for certain subsets S of S_n , the eigenvalues of $\Gamma(S_n, S)$ are integers.

1.2. Arrangement graphs

For $k \leq n$, a k -permutation of $[n]$ is an injective function from $[k]$ to $[n]$. So any k -permutation π can be represented by a vector (i_1, \dots, i_k) where $\pi(j) = i_j$ for $j = 1, \dots, k$. Let $1 \leq r \leq k \leq n$. The (n, k, r) -arrangement graph $A(n, k, r)$ has all the k -permutations of $[n]$ as vertices and two k -permutations are adjacent if they differ in exactly r positions. Formally, the vertex set $V(n, k)$ and edge set $E(n, k, r)$ of $A(n, k, r)$ are

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