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Some weighted Bartholdi zeta function of a digraph

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ABSTRACT

Related to the weighted bond scattering matrix of a graph, we define some weighted Bartholdi zeta function of a digraph D, and give a determinant expression for it. We present a decomposition formula for the weighted zeta function of a group covering of D. Furthermore, we define an L-function of D, and give a determinant expression for it. As a corollary, we express the weighted Bartholdi zeta function of a group covering of D by means of its L-functions.

1. Introduction

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [10]. In [10], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [17,18]. Hashimoto [9] generalized Ihara's result on the zeta function of a regular graph to an irregular graph,

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and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [3] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

For two variable zeta function of a graph, Bartholdi [2] defined and gave a determinant expression of the Bartholdi zeta function of a graph. Mizuno and Sato [13] considered a new zeta function of a digraph, and defined a new zeta function of a digraph by using not an infinite product but a determinant.

The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph (see [4,6,8,16]). Smilansky [16] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs, and expressed some determinant on the bond scattering matrix of a graph G by using the characteristic polynomial of its Laplacian. Furthermore, Smilansky [16] considered a weighted version of the determinant on the bond scattering matrix of a graph G, and expressed it by using the characteristic polynomial of the weighted Laplacian of G. Oren, Godel and Smilansky [14] defined a new Bartholdi zeta function of a graph related to its weighted scattering matrix.

In this paper, we generalize the above zeta function of a digraph and the above Bartholdi zeta function of a graph. We define a new weighted Bartholdi zeta function of a digraph with respect to a vertex weight and an arc weight, and present its determinant expression. Furthermore, we present a decomposition formula for the new weighted Bartholdi zeta function of a group covering of a digraph D, and a determinant expression for the new weighted Bartholdi L-function of D.

2. Preliminaries

Graphs and digraphs treated here are finite. Let $G=(V(G),E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. Furthermore, let $D=(V(D),A(D))$ be a connected digraph (possibly multiple arcs and loops) with the set $V(D)$ of vertices and the set $A(D)$ of oriented edges. For $u,v\in V(D)$, an arc (u,v) is the oriented edge from u to v. For a graph G, set $D(G)=\{(u,v),(v,u) \mid uv\in E(G)\}$. A digraph $D_G=(V(G),D(G))$ is called the *symmetric digraph* corresponding to G. For $e=(u,v)\in D(G)$, set $u=o(e)$ and $v=t(e)$. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

Let G be a connected graph and D a connected digraph. A *path P of length n* in G (or D) is a sequence $P=(e_1,\dots,e_n)$ of n arcs such that $e_i\in D(G)$ (or $A(D)$), $t(e_i)=o(e_{i+1})$ $(1\leqslant i\leqslant n-1)$, where indices are treated mod n. Set $|P|=n$, $o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called an *$(o(P),t(P))$-path*. We say that a path $P=(e_1,\dots,e_n)$ has a *backtracking* or *bump* if $e_{i+1}=e_i^{-1}$ for some i $(1\leqslant i\leqslant n-1)$. A (v,w)-path is called a *v-cycle* (or *v-closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(e_1,\dots,e_n)$ is the cycle $C^{-1}=(e_n^{-1},\dots,e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1=(e_1,\dots,e_m)$ and $C_2=(f_1,\dots,f_m)$ are called *equivalent* if there exists k such that $f_j=e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. The *cyclic bump count* $cbc(C)$ of a cycle $C=(e_1,\dots,e_n)$ in G (or D) is

$$cbc(C)=\left|\left\{i=1,\dots,n \,\middle|\, e_{i+1}=e_i^{-1}\right\}\right|,$$

where $e_{n+1}=e_1$.

Now, we state a new zeta function of a digraph by Mizuno and Sato [13].

Let D be a connected digraph with n vertices $v_1,\dots,v_n$ and m arcs. Then we consider an $n\times n$ matrix $\mathbf{W}=\mathbf{W}(D)=(w_{ij})_{1\leqslant i,j\leqslant n}$ with ij entry the complex variable w_{ij} if $(v_i,v_j)\in A(D)$, and $w_{ij}=0$ otherwise. The matrix $\mathbf{W}=\mathbf{W}(D)$ is called the *weighted matrix* of D. Furthermore, let $w(v_i,v_j)=w_{ij}$, $v_i,v_j\in V(D)$ and $w(e)=w_{ij}$, $e=(v_i,v_j)\in A(D)$. Then $w: A(D)\longrightarrow \mathbf{C}$ is called a *weight* of D. For each path $P=(e_1,\dots,e_r)$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P)=w(e_1)\cdots w(e_r)$.

Let D be a connected digraph with n vertices and m arcs, and $\mathbf{W}=\mathbf{W}(D)$ a weighted matrix of D. Two $m\times m$ matrices $\mathbf{B}=\mathbf{B}(D)=(\mathbf{B}_{e,f})_{e,f\in A(D)}$ and $\mathbf{J}=\mathbf{J}(D)=(\mathbf{J}_{e,f})_{e,f\in A(D)}$ are defined as follows: