Linear Algebra and its Applications 445 (2014) 181–205

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the symmetric doubly stochastic inverse eigenvalue problem ☆

Ying-Jie Lei [a], Wei-Ru Xu [a,*], Yong Lu [a], Yan-Ru Niu [a], Xian-Ming Gu [b]

[a] School of Science, North University of China, Taiyuan, 030051, China
[b] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 611731, China

A R T I C L E   I N F O

Article history:
Received 1 July 2013
Accepted 4 December 2013
Available online 22 December 2013
Submitted by R. Loewy

MSC:
65F18
15A51
15A18
15A12

Keywords:
Inverse eigenvalue problem
Symmetric doubly stochastic matrix
Symmetric positive doubly stochastic matrix
Sufficient condition
Convexity

A B S T R A C T

Let $\sigma = (1, \lambda_2, \dots, \lambda_n)$ be a list of real numbers. The symmetric doubly stochastic inverse eigenvalue problem (hereafter SDIEP) is to determine the list σ which can occur as the spectrum of an $n \times n$ symmetric doubly stochastic matrix A. If the matrix A is positive, we can necessarily obtain a subproblem, symmetric positive doubly stochastic inverse eigenvalue problem (hereafter SPDIEP), of the SDIEP. In this paper, we give some sufficient conditions for the SDIEP and SPDIEP and prove that the set formed by the spectra of all $n \times n$ symmetric positive doubly stochastic matrices is non-convex for $n \geqslant 4$.

☆ Supported by the National Natural Science Foundation of China (Grant No. 11071227).
* Corresponding author.
E-mail addresses: fl3921767@nuc.edu.cn (Y.-J. Lei), weiruxu@foxmail.com (W.-R. Xu).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.12.005

1. Introduction

A real square matrix with nonnegative entries all of whose row sums or column sums are equal to 1 is referred to as stochastic. Moreover, if all of its row sums and column sums are equal to 1, then it is said to be doubly stochastic. A real matrix $A = (a_{ij})_{n\times n}$ is said to be generalized symmetric doubly stochastic if it is symmetric and all its row sums and column sums are the same constant, say α, i.e.

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = \alpha, \quad i, j = 1, \dots, n.$$

The theory of doubly stochastic matrices has been the subject of numerous investigations as it arises in a variety of applications in different fields, such as probability and statistics, communication theory, combinatorics and graph theory (see, e.g., [1,2,8,13]), etc. For this theory, there exists an interesting and challenging research topic: SDIEP. It is a subproblem of the doubly stochastic inverse eigenvalue problem, which asks the necessary and sufficient conditions for a list $\sigma = (1, \lambda_2, \dots, \lambda_n)$ of complex numbers to be the spectrum of an $n \times n$ doubly stochastic matrix. All the above problems have drawn considerable interest (see, [2,3,5–7,10–12,14–32] and the references therein). So far, the SDIEP has only been solved for the case $n = 3$ by Perfect and Mirsky [21]. The case $n = 4$ for symmetric doubly stochastic matrices of trace zero has also been solved by them in the above reference. Subsequently, Mourad also solved the above case $n = 4$ by a mapping and convexity technique in [16]. In this reference, the case $n = 4$ with trace two is also solved. But a complete characterization is unknown for all real n-tuples, the problem still remains open for the cases $n = 4$ with other nonzero trace and $n \geqslant 5$. The difficulty lies in deriving necessary and sufficient conditions to complete this conundrum, even if many sufficient conditions have appeared.

Until now, there are four methods to solve the SDIEP, i.e. constructing an idempotent system [21], establishing Soules basis [27], Householder transformation [32] and fast Fourier transformation [25,26]. As a matter of fact, for a real n-tuple $\sigma = (1, \lambda_2, \dots, \lambda_n)$ all the methods require us to establish an $n \times n$ nonsingular matrix P and explore the conditions under which $A = P^{-1}\Lambda P \geqslant 0$ and can be symmetric doubly stochastic with $\Lambda = \operatorname{diag}(1, \lambda_2, \dots, \lambda_n)$. The Soules basis was first adopted by Elsner, et al. [6]. Subsequently, in [12] and [20] the authors, via using Soules basis, discussed the convexity of the set $\mathcal{R}_n$ of n-tuples of realizable spectra of nonnegative symmetric matrices. In 2009, Zhu, et al. [32] proposed the disadvantages of the Soules basis and fast Fourier transformation and employed a new algorithm, Householder transformation. In 2012, Mourad et al. [17] developed a novel algorithm to solve the SDIEP through constructing a nonsingular matrix by Soules basis. But all the methods only apply to their respective feasible region, not to the general cases. So this is the reason why the SDIEP has not been solved completely.