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Perturbations on constructible lists preserving realizability in the NIEP and questions of Monov

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ABSTRACT

In [5] the authors showed that if $\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2}, \lambda_4, \ldots, \lambda_n)$ is realizable where λ_1 is the Perron eigenvalue and λ_2 is non-real, then so too is $\sigma = (\lambda_1 + 4t, \lambda_2 + t, \overline{\lambda_2} + t, \lambda_4, \ldots, \lambda_n)$. They asked if 4 can be replaced by 1 or 2 or what is the least possible multiple $c \geqslant 1$ of t in order for this perturbation to be realizable. In [2] the authors showed that for $n = 4$ one can find certain spectra for which the result holds when $c = 1$ provided t is "small". In this paper we show that $c = 2$ is best possible and we construct a realizing matrix for $c = 2$ when t is sufficiently small. We also address some questions of Monov concerning the realizability of the derivative of a realizable polynomial and if such a polynomial must have positive power sums.

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for a complete set of necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ of complex numbers so that this list be the set of eigenvalues of an $n \times n$ entrywise nonnegative matrix A. If

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we restrict the list σ to be real then the problem is called the real nonnegative inverse eigenvalue problem (RNIEP). The list σ is said to be *realizable* if there exists such an entrywise nonnegative matrix A of order n with spectrum σ. Also, we say that the polynomial $f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$ is realizable when such an A has $f(x)$ as its characteristic polynomial. If $\rho = \max\{|\lambda_j|\colon j = 1, 2, \ldots, n\}$, then ρ is an eigenvalue of A and there exists an eigenvector $v \geqslant 0$ with $Av = \rho v$. We call ρ and v the *Perron root* and *Perron eigenvector* respectively. An obvious necessary condition for the NIEP is that the traces of the powers of the realizing matrix A must be nonnegative, hence

$$s_k := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k \geqslant 0.$$

A stronger condition found independently by Loewy and London [10], and Johnson [6], known as the JLL inequalities state that

$$n^{m-1} s_{km} \geqslant s_k^m \quad \text{for } m, k = 1, 2, \ldots$$

if σ is the spectrum of a nonnegative matrix. An interesting problem in this area is as follows:

Given a realizable list $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ and a realizing matrix A, what perturbations can we perform on the list σ which will again yield a realizable list σ' with realizing matrix A'? In [1], Brauer (Theorem 1) showed that if $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is realizable then $\sigma = (\lambda_1 + t, \lambda_2, \lambda_3, \ldots, \lambda_n)$ is also realizable for all $t > 0$, where λ_1 represents the Perron root. The proof uses the Perron eigenvector v to construct a rank-one perturbation $B = A + tvu^t$ where $u \geqslant 0$ and $u^t v = 1$. Perfect [13] extended this result to show how to change r eigenvalues of an $n \times n$ nonnegative matrix with $r < n$ without changing the remaining $n - r$ eigenvalues. This result uses a rank-r perturbation again using the r eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_r$. Rojo and Soto [15] extended these results further to find a new realizability criterion for the RNIEP.

2. Perturbation results

Of more immediate relevance to this discussion are the perturbation results of Wuwen Guo [4].

In Theorem 3.1 [4] he proves

Theorem 1. *Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ be realizable by a nonnegative matrix, where λ_1 is the Perron root and λ_2 is real. Then, for all $t \geqslant 0$, the list $(\lambda_1 + t, \lambda_2 + \varepsilon t, \lambda_3, \ldots, \lambda_n)$ is also realizable for all $\varepsilon \in [-1, 1]$.*

Laffey, in Theorem 1.1 [7] (see [5] for an alternative proof) proves an analogue of Guo's theorem in the non-real case by showing