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Fixed points of functions with max-weighted quasi-arithmetic mean operator

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Let $\lambda \in (0,1)$ and f be a continuous, strictly monotone real-valued function. The weighted quasi-arithmetic mean of two numbers a, b is defined by $a \otimes b = f^{-1}(\lambda f(a) + (1-\lambda)f(b))$. Let $A = [a_{ij}]$ be an $n \times n$ real matrix and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. We construct a function $\psi^{(A)} = (\psi_1^{(A)}, \psi_2^{(A)}, \ldots, \psi_n^{(A)}) : \mathbb{R}^n \to \mathbb{R}^n$ by $\psi_j^{(A)}(x) = \max_{1 \leqslant l \leqslant n}\{x_l \otimes a_{lj}\}$ for all $j = 1, 2, \ldots, n$. In this paper we show that $\psi^{(A)}$ has a unique fixed point $\hat{x}^{(A)}$. Moreover, it can be shown that for each $x \in \mathbb{R}^n$ the sequence $\{x^{(A,k)}\}$, generated by the following iterative scheme: $x^{(A,0)} = x$ and $x^{(A,k)} = \psi^{(A)}(x^{(A,k-1)})$ for all $k \geqslant 1$, converges to the unique fixed point $\hat{x}^{(A)}$. Besides, some properties of the fixed point are derived. As an application, our results imply that the max-weighted quasi-arithmetic mean powers of any matrix are always convergent. The continuity of the function $\eta : M_{n \times n}(\mathbb{R}) \to \mathbb{R}^n$ defined by $\eta(A) = \hat{x}^{(A)}$ is proposed as well.

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1. Introduction

A function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *mean* if

$$\min\{a, b\} \leqslant \varphi(a, b) \leqslant \max\{a, b\} \quad \text{for all } a, b \in \mathbb{R}. \tag{1}$$

A mean φ is called *strict* if these inequalities in (1) are strict for all $a \neq b$; and *increasing* if it is increasing with respect to each of the variables. It is obvious that an increasing function φ is a mean if and only if $\varphi(a, a) = a$ for all $a \in \mathbb{R}$ (see, e.g. [20]).

An important class of means that includes the arithmetic, geometric, and harmonic means, is described as follows: A function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *weighted quasi-arithmetic mean* if there exists a continuous and strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda \in (0, 1)$ such that

$$\varphi(a, b) = f^{-1}\big(\lambda f(a) + (1 - \lambda) f(b)\big) \quad \text{for all } a, b \in \mathbb{R},$$

where f^{-1} is the inverse function of f. In this case, we say that φ is generated by f and λ. We note that if $\lambda = \frac{1}{2}$ then the weighted quasi-arithmetic mean is the so-called *quasi-arithmetic mean* (see, e.g. [9,19]). It is easy to verify that a weighted quasi-arithmetic mean φ is a strict and increasing mean. For our convenience, we shall use the notation $\otimes$ to represent the operator deduced by the weighted quasi-arithmetic mean φ generated by f and λ, that is,

$$a \otimes b = \varphi(a, b) = f^{-1}\big(\lambda f(a) + (1 - \lambda) f(b)\big) \quad \text{for all } a, b \in \mathbb{R}.$$

We note that $(a \otimes b) \otimes c$ is not necessary equal to $a \otimes (b \otimes c)$, that is, the operation $\otimes$ is nonassociative. Hence, for $k \geqslant 2$ the product $a_1 \otimes a_2 \otimes \cdots \otimes a_k$ is defined by

$$a_1 \otimes a_2 \otimes \cdots \otimes a_k = \Big(\Big(\big((a_1 \otimes a_2) \otimes a_3\big) \otimes \cdots\Big) \otimes a_k\Big)$$

for all $a_1, a_2, \ldots, a_k \in \mathbb{R}$. It is easy to see that

$$\begin{aligned} &a_1 \otimes a_2 \otimes \cdots \otimes a_k \\ &\quad = f^{-1}\big(\lambda^{k-1} f(a_1) + \lambda^{k-2}(1 - \lambda) f(a_2) + \cdots + \lambda(1 - \lambda) f(a_{k-1}) + (1 - \lambda) f(a_k)\big). \end{aligned}$$

Let $M_{n\times n}(\mathbb{R})$ be the set of all $n \times n$ real matrices and $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n)^T: x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$. For a matrix $A = [a_{ij}]$, we also denote a_{ij} by $[A]_{ij}$. For each $A \in M_{n\times n}(\mathbb{R})$, we define a function $\psi^{(A)} = (\psi_1^{(A)}, \psi_2^{(A)}, \ldots, \psi_n^{(A)}) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi_j^{(A)}(x) = \max_{1\leqslant l\leqslant n} \{x_l \otimes a_{lj}\} = \max_{1\leqslant l\leqslant n} f^{-1}\big(\lambda f(x_l) + (1 - \lambda) f(a_{lj})\big), \quad j = 1, 2, \ldots, n. \tag{2}$$

Since f and f^{-1} are continuous functions, we see that $\psi^{(A)}$ is continuous on $\mathbb{R}^n$. A vector $x \in \mathbb{R}^n$ is said to be a fixed point of $\psi^{(A)}$ if $\psi^{(A)}(x) = x$. In this paper, we show that