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Generalized derivations on unital algebras determined by action on zero products

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A B S T R A C T

Let $\mathcal{A}$ be a unital algebra having a nontrivial idempotent and let $\mathcal{M}$ be a unitary $\mathcal{A}$-bimodule. We consider linear maps $F, G : \mathcal{A} \to \mathcal{M}$ satisfying $F(x)y + xG(y) = 0$ whenever $x, y \in \mathcal{A}$ are such that $xy = 0$. For example, when $\mathcal{A}$ is zero product determined algebra (e.g. algebra generated by idempotents) F and G are generalized derivations $F(x) = F(1)x + D(x)$ and $G(x) = xG(1) + D(x)$ for all $x \in \mathcal{A}$, where $D : \mathcal{A} \to \mathcal{M}$ is a derivation. If $\mathcal{A}$ is not generated by idempotents then there exist also nonstandard solutions for maps F and G. In the case of $\mathcal{A}$ being a triangular algebra under some condition on bimodule $\mathcal{M}$ the characterization of maps F and G is given. We also consider conditions on algebra $\mathcal{A}$ making it a zero product determined algebra.

1. Introduction

Throughout the paper let R be a commutative ring with an identity, let $\mathcal{A}$ be a unital algebra over R and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Let us assume that $\mathcal{A}$ has an idempotent $e \neq 0, 1$ and let f denote the idempotent $1 - e$. In this case $\mathcal{A}$ can be

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represented in the so-called Peirce decomposition form $\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f$, where $e\mathcal{A}e$ and $f\mathcal{A}f$ are subalgebras with unitary elements e and f, respectively, $e\mathcal{A}f$ is an $(e\mathcal{A}e, f\mathcal{A}f)$-bimodule and $f\mathcal{A}e$ is an $(f\mathcal{A}f, e\mathcal{A}e)$-bimodule. If $f\mathcal{A}e = \{0\}$ then $\mathcal{A}$ is a triangular algebra.

Recall that *a generalized derivation* is a linear map $F : \mathcal{A} \to \mathcal{M}$ with a corresponding linear map $G : \mathcal{A} \to \mathcal{M}$ satisfying $F(xy) = F(x)y + xG(y)$ for all $x, y \in \mathcal{A}$. Examples of generalized derivations are *derivations*, i.e. linear maps satisfying $F(xy) = F(x)y + xF(y)$, *left multipliers*, i.e. $F(xy) = F(x)y$, in the case of unital algebra $\mathcal{A}$ also *right multipliers*, i.e. $F(xy) = xF(y)$, and the sums of these maps. Our main goal is to characterize generalized derivations determined by action on zero product elements on algebras containing a nontrivial idempotent, i.e. linear maps $F, G : \mathcal{A} \to \mathcal{M}$ satisfying

$$x, y \in \mathcal{A}, \quad xy = 0 \quad \Longrightarrow \quad F(x)y + xG(y) = 0. \tag{1}$$

The motivation for our research are articles [2,4,7,11–14] where the derivations determined by action on zero product elements in unitary algebras containing a nontrivial idempotent are studied. Under some conditions on algebra $\mathcal{A}$ the map $F : \mathcal{A} \to \mathcal{A}$ satisfying $F(x)y + xF(y) = 0$ whenever $xy = 0$ is of the form $F(x) = F(1)x + D(x)$ for all $x \in \mathcal{A}$, with D being a derivation and $F(1) \in Z(\mathcal{A})$.

In [4, Theorem 4.4] Matej Brešar showed by using a more general approach that in the case of $\mathcal{A}$ being a unital algebra generated by idempotents every map $F : \mathcal{A} \to \mathcal{M}$ satisfying $F(x)y + xF(y) = 0$ whenever $xy = 0$ is of the standard form $F(x) = F(1)x + D(x)$, where D is a derivation and $F(1) \in Z(\mathcal{M})$.

What is the standard solution of our generalized problem (1)? If $\mathcal{A}$ is a zero product determined algebra (e.g. algebra generated by idempotents) then F and G are generalized derivations of the form

$$F(x) = F(1)x + D(x) \quad \text{and} \quad G(x) = xG(1) + D(x) \tag{2}$$

for all $x \in \mathcal{A}$, where $D : \mathcal{A} \to \mathcal{M}$ is a derivation. If $\mathcal{A}$ is not a zero product determined algebra there can also exist nonstandard solutions for (1).

Let us point out the result by Tsiu-Kwen Lee in [14]. Let $\mathcal{R}$ be a prime ring whose symmetric Martindale quotient ring contains a nontrivial idempotent. He characterized generalized skew derivations on $\mathcal{R}$ by acting on zero products. More precise, he characterized additive maps $F, G : \mathcal{R} \to \mathcal{R}$ such that $F(x)y + \sigma(x)G(y) = 0$ for all $x, y \in \mathcal{R}$ with $xy = 0$, where $\sigma : \mathcal{R} \to \mathcal{R}$ is an automorphism. From the main result [14, Theorem 1.1] it follows that if $\mathcal{A}$ is a unital prime algebra containing a nontrivial idempotent and linear maps $F, G : \mathcal{A} \to \mathcal{A}$ satisfy (1) then F, G are of the form (2).

In the second section we examine the conditions under which a unital algebra containing a nontrivial idempotent is zero product determined. We give basic remarks about this problem, some of them are already known from [3–5], and in Example 1 we introduce