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Fredholm theory for band-dominated and related operators: A survey

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ABSTRACT

This paper presents the Fredholm theory on l^p-spaces for band-dominated operators and important subclasses, such as operators in the Wiener algebra. It particularly closes several gaps in the previously known results for the case $p = \infty$ and addresses the open questions raised by Chandler-Wilde and Lindner in [S.N. Chandler-Wilde, M. Lindner, Limit operators, collective compactness and the spectral theory of infinite matrices, Mem. Amer. Math. Soc. 210 (989) (2011)]. The main tools are provided by the limit operator method and an algebraic framework for the description and adaption of Fredholmness and convergence. A comprehensive overview of this approach is given.

1. Introduction

During the last years a far reaching theory on band-dominated operators grew up and revealed deep results concerning their Fredholm property, their spectral properties, the applicability (and further side effects) of the finite section method, and was successfully applied to several more concrete subclasses of operators having more advanced properties. Among them one can find Toeplitz, Hankel or Jacobi operators, hence also discrete Schroedinger operators.

Besides, the wonderful concept of operator algebras arising from approximate projections and the limit operator method were developed. Today one may say that large parts of this theory are very well understood and finalized. However, there are some results which still require some additional assumptions that seem to be redundant but could not be removed completely yet. One prominent example is the need for a predual setting in the case of band-dominated operators on l^∞-spaces in

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the works of Lindner. A collection of eight open problems was stated by Chandler-Wilde and Lindner in the final chapter of [3].

The aim of the present text is to give an overview on the latest state of the art and to close several gaps in the theory on band-dominated operators. We show that a couple of results being known for band-dominated operators actually hold in a more general context, and we try to clear up some possible generalizations which have already been indicated in the literature. We particularly contribute to seven of the eight open questions stated in [3] answering four of them completely.

1.1. Band-dominated operators

Let us start introducing the basic notions and the precise description of two of the mentioned (and actually redundant) conditions (1) and (2).

Sequence spaces. Given $N \in \mathbb{N}$, a Banach space X, and the parameter $1 \leqslant p < \infty$ we let $l^p = l^p(\mathbb{Z}^N, X)$ denote the space of all functions $x : \mathbb{Z}^N \to X$ with the property

$$\|x\|_p := \left(\sum_{i \in \mathbb{Z}^N} \|x(i)\|^p \right)^{1/p} < \infty.$$

Provided with the norm $\|x\|_p$, l^p becomes a Banach space. It is convenient to refer to such functions as sequences $(x_i)_{i \in \mathbb{Z}^N}$ with $x_i = x(i)$. The above condition then simply means that the sequences shall be p-summable. Analogously, one introduces the respective Banach spaces $l^\infty = l^\infty(\mathbb{Z}^N, X)$ of all bounded sequences $x = (x_i)$ of elements $x_i \in X$ with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty := \sup\{\|x_i\| : i \in \mathbb{Z}^N\}$$

and, finally, its closed subspace $l^0 = l^0(\mathbb{Z}^N, X)$ consisting of all bounded sequences (x_i) with $\|x_i\| \to 0$ as $|i| \to \infty$.

Note that, choosing $X = L^p((0,1)^N)$, $l^p(\mathbb{Z}^N, X)$ is isometrically isomorphic to $L^p(\mathbb{R}^N)$. Hence, all subsequent definitions and results for the "discrete" l^p-cases have their "continuous" L^p-counterparts. This is a well known and frequently utilized observation (see e.g. [12,15,9,13,17,8,3]), and we will not go into further details here.

The operators. Every sequence $a = (a_i) \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$, where $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on X, gives rise to a bounded linear operator aI of multiplication on each of the spaces l^p by $(x_i) \mapsto (a_i x_i)$.

Another basic family of operators in $\mathcal{L}(X)$ is given by the shifts

$$V_\alpha : l^p \to l^p, \quad (x_i) \mapsto (x_{i-\alpha}) \quad (\alpha \in \mathbb{Z}^N).$$

This is everything we need for the definition of band-dominated operators:

Definition 1. A finite sum of the form $\sum_\alpha a_\alpha V_\alpha$ is called a band operator. The limits of sequences of band operators, taken w.r.t. the operator norm, are said to be band-dominated, and the set of all band-dominated operators on l^p shall be denoted by $\mathcal{A}_p$.

Clearly, band operators form a (non-closed) algebra $\mathcal{B}$ of bounded linear operators on each of the spaces l^p, $p \in \{0\} \cup [1, \infty]$. Thus, $\mathcal{A}_p$ is a Banach algebra for every p. Note that $\mathcal{B}$ is independent of the choice of the parameter p, whereas $\mathcal{A}_p$ depends on p.

For a nice introduction and a comprehensive discussion we refer to the work of Lindner (e.g. [8,3]) and the book of Rabinovich, Roch and Silbermann [17].

1.2. The Wiener algebra

A prominent algebra which is somehow between $\mathcal{B}$ and $\mathcal{A}_p$ and which provides a remarkable Fredholm behavior is the so-called Wiener algebra $\mathcal{W}$: For band operators $A = \sum_\alpha a_\alpha V_\alpha$ set