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Smith forms for adjacency matrices of circulant graphs

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ABSTRACT

We calculate the Smith normal form of the adjacency matrix of each of the following graphs or their complements (or both): complete graph, cycle graph, square of the cycle, power graph of the cycle, distance matrix graph of cycle, Andrásfai graph, Doob graph, cocktail party graph, crown graph, prism graph, Möbius ladder. The proofs operate by finding the abelianization of a cyclically presented group whose relation matrix is column equivalent to the required adjacency matrix.

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1. Introduction

The *circulant matrix* $\mathrm{circ}_n(a_0, \dots, a_{n-1})$ is the $n \times n$ matrix whose first row is $(a_0, \dots, a_{n-1})$ and where row $(i+1)$ $(0 \leqslant i \leqslant n-2)$ is a cyclic shift of row i by one column. A *circulant graph* is a graph that is isomorphic to a graph whose adjacency matrix is circulant. We shall write $A(\Gamma)$ for the adjacency matrix of a graph Γ. Given graphs Γ, Γ' if $\det(A(\Gamma)) \neq \det(A(\Gamma'))$ (in particular if precisely one of $A(\Gamma)$, $A(\Gamma')$ is singular) then Γ, Γ' are non-isomorphic. Similarly, if $\mathrm{rank}(A(\Gamma)) \neq \mathrm{rank}(A(\Gamma'))$ then Γ, Γ' are non-isomorphic. Singularity, rank, and determinants of various families of circulant graphs are considered (for example) in [3,7,16,23].

For an $n \times n$ integer matrix M, the *Smith normal form* of M, written $\mathrm{SNF}(M)$, is the $n \times n$ diagonal integer matrix

$$S = \mathrm{diag}_n(d_0, \dots, d_{n-1})$$

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where $d_0, \ldots, d_{n-1} \in \mathbb{N} \cup \{0\}$ and $d_i | d_{i+1}$ $(0 \leqslant i \leqslant n-2)$ is such that there exist invertible integer matrices P, Q such that $PMQ = S$. The matrix S is unique and may be obtained from M using (integer) elementary row and column operations. Given graphs Γ, Γ' if $\mathrm{SNF}(A(\Gamma)) \neq \mathrm{SNF}(A(\Gamma'))$ then Γ, Γ' are non-isomorphic. Since also $\det(\mathrm{SNF}(A(\Gamma))) = |\det(A(\Gamma))|$ and $\mathrm{rank}(\mathrm{SNF}(A(\Gamma))) = \mathrm{rank}(A(\Gamma))$ the Smith normal form is a more refined invariant than (the absolute value of) the determinant and the rank.

The purpose of this article is to calculate the Smith normal form for the adjacency matrices of various families of circulant graphs. The calculations of determinant, rank and determination of singularity of adjacency matrices of circulant graphs cited above use a well known number theoretic expression for the eigenvalues of a circulant matrix and a graph theoretic formula for the determinant of the adjacency matrix of a graph, due to Harary [5, Proposition 7.2]. Our methods use techniques from combinatorial group theory, namely the application of Tietze transformations (the addition or removal of generators or relations to or from the presentation while leaving the group unchanged) on cyclic presentations. We now outline our method of proof; for further background we refer the reader to [14].

Let $P = \langle x_0, \ldots, x_{n-1} \mid r_0, \ldots, r_{m-1} \rangle$ be a group presentation defining a group G. For $0 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m-1$ let a_{ij} denote the exponent sum of generator x_i in relator r_j (that is, the number of times x_i appears in r_j, counting multiplicities and sign). The matrix $M = (a_{ij})_{n \times m}$ is called the *relation matrix* of P. The abelianization G^{ab} of G can be written in the form

$$\mathbb{Z}_{\delta_0} \oplus \cdots \oplus \mathbb{Z}_{\delta_{p-1}} \oplus \mathbb{Z}^q \tag{1}$$

where $1 < \delta_0 | \delta_1 | \cdots | \delta_{p-1}$ and $p + q \leqslant n$. These numbers determine the non-unity diagonal entries of the Smith normal form for the relation matrix of any presentation of G; in particular we have

$$\mathrm{SNF}(M) = \mathrm{diag}_n(1, \ldots, 1, \delta_0, \ldots, \delta_{p-1}, \underbrace{0, \ldots, 0}_{q}). \tag{2}$$

Let $w = w(x_0, \ldots, x_{n-1})$ be a word in generators $x_0, \ldots, x_{n-1}$. The *cyclically presented group* $G_n(w)$ is the group defined by the *cyclic presentation*

$$P_n(w) = \langle x_0, \ldots, x_{n-1} \mid w_0, \ldots, w_{n-1} \rangle$$

where $w_i = w(x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+(n-1)})$ (subscripts mod n). The relation matrix of $P_n(w)$ is the circulant matrix $\mathrm{circ}_n(a_0, \ldots, a_{n-1})$ where for each $0 \leqslant i \leqslant n-1$ the entry a_i is the exponent sum of x_i in $w(x_0, \ldots, x_{n-1})$. Thus our method of proof is as follows. Given a circulant matrix $C = \mathrm{circ}_n(a_0, \ldots, a_{n-1})$ we define a suitable word $w(x_0, \ldots, x_{n-1})$ so that the exponent sum of x_i in $w(x_0, \ldots, x_{n-1})$ is a_i $(0 \leqslant i \leqslant n-1)$. We use Tietze transformations to simplify the presentation to obtain the abelianization G^{ab} in the form (1) and hence obtain $\mathrm{SNF}(C)$.

In principle, our proofs can be written purely in terms of elementary row and column operations, but cyclically presented groups provide a convenient framework and language within which to operate. We digress from this method of proof in Section 5 where we make use of multiplicativity results of Newman [17]. In Table 1 we summarize the graphs and adjacency matrices (up to column equivalence), and point to the result which gives the Smith form. (We write $\overline{\Gamma}$ to denote the complement of a graph Γ.)

2. Circulant graphs

In this section we review the definitions of various families of circulant graphs (the webpage [25] is a useful resource for such information) and, where necessary and possible, relabel vertices so that their adjacency matrices are column equivalent to a matrix of one of the following forms:

$$F_{n,s} = \mathrm{circ}_n(\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{n-s}) \quad (1 \leqslant s \leqslant n),$$

$$F_{n,s,r} = \mathrm{circ}_n(\underbrace{v, \ldots, v}_{s}, \underbrace{0, \ldots, 0}_{n-rs}) \quad \bigl(rs - (r-1) \leqslant n,\ 1 \leqslant r\bigr),$$

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