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On the distribution of eigenspaces in classical groups over finite rings

Michael Adam

TU Kaiserslautern, Germany

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ABSTRACT

Jeffrey D. Achter establishes in [2] a connection between the distribution of class groups of function fields and the distribution of eigenspaces in symplectic groups. Gunter Malle relates this idea in [10] to the number field case. Motivated by these texts we compute here for a finite $\mathfrak{p}$-torsion R-module H, where R is a Dedekind ring with finite quotients and $q = |R/\mathfrak{p}|$, the limit

$$\begin{aligned} P_{G,\infty,q,f}(H) &:= \lim_{n\to\infty} P_{G,n,q,f}(H) \\ &:= \lim_{n\to\infty} \frac{|\{g \in G_n(R/\mathfrak{p}^f) \mid \ker(g-\mathbf{1}) \cong H\}|}{|G_n(R/\mathfrak{p}^f)|} \end{aligned}$$

for certain classical groups G of increasing dimension n. In doing so we extend the results of Jason Fulman ([4] and [5]) concerning distributions of eigenspaces over finite fields. Furthermore we give a reasonable backup for the conjecture of G. Malle (2.1 in [10]).

1. Introduction

J. Fulman, Peter M. Neumann and Cheryl E. Praeger consider in [7], [4], [5] and [6] natural questions in pure mathematics related to the probability that an element of a matrix group has a certain property. For instance J. Fulman determines in [4] the proportion of elements $g \in \mathrm{GL}_n(\mathbb{F}_q)$ having a k-dimensional kernel and in [5] he shows the corresponding result for the symplectic groups.

J. Achter proves in [2] that there is a close relation between the distribution of class groups of function fields and the distribution of the eigenspaces corresponding to the eigenvalue one in symplectic groups. In [10] G. Malle extends this idea to the number field case.

E-mail address: adam@mathematik.uni-kl.de.

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Motivated by work of these authors the purpose of this text is to generalize Fulman's computations done over finite fields to the case of finite rings and as an application to compute the following limit

$$P_{\mathrm{Sp}^{(1)},\infty,q,f}(H) := \lim_{n\to\infty} P_{\mathrm{Sp}^{(1)},n,q,f}(H) := \frac{|\{g \in \mathrm{Sp}_{2n}^{(1)}(R/\mathfrak{p}^f) \mid \ker(g-\mathbf{1}) \cong H\}|}{|\mathrm{Sp}_{2n}^{(1)}(R/\mathfrak{p}^f)|}$$

for a finite R-module H where $\mathrm{Sp}_{2n}^{(1)}(R/\mathfrak{p}^f) := \{g \in \mathrm{GL}_{2n}(R/\mathfrak{p}^f) \mid g^t J_n g \equiv J_n \mod \mathfrak{p}\}$ and J_n defines an alternating form. This will be related to the distribution of the p-part of class groups $\mathrm{Cl}(K/K_0)$ for a number field K_0 containing the pth but not the p^2rd roots of unity (see Theorem 3.23 and Remark 3.24).

In the first section of this paper we collect the essential notations and required results from module theory and also about classical groups. The second section is divided into three parts, each of which discusses one of the types of classical groups we deal with. In the first part we compute $P_{\mathrm{GL},\infty,q,f}(H)$ (see Theorem 3.8), in the second we determine $P_{\mathrm{Sp},\infty,q,f}(H)$ (see Theorem 3.19) and in the last, as mentioned above, we calculate $P_{\mathrm{Sp}^{(1)},\infty,q,f}(H)$ (see Theorem 3.23).

This is part of the authors dissertation at TU Kaiserslautern and the author thanks Prof. Dr. Gunter Malle for helpful ideas and comments on this text.

2. Preliminaries

In this section we introduce some notations and recall some results concerning modules over Dedekind domains. Moreover we present formulas for the order of classical groups over finite rings.

2.1. Notation

Notation 2.1. The set of positive numbers (without zero) is denoted by $\mathbb{N}$. For the set of positive primes we write $\mathbb{P}$. By a ring we always understand a commutative ring with identity and we write $R^\times$ for the unit group of the ring R. The cyclic group of order p^α, for $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$, we denote by $\mathbb{Z}/p^\alpha\mathbb{Z}$ and also the corresponding quotient ring. We write $\mathrm{Mat}_n(R)$ for the set of all square matrices of dimension n with entries in R. For an element $g \in \mathrm{Mat}_n(R)$ we denote by g^t its transpose. For natural numbers n and k we set

$$(n)_k := \prod_{i=1}^{k}(1-n^{-i})$$

and

$$(n)_\infty := \prod_{i=1}^{\infty}(1-n^{-i}).$$

Note that this product converges for all $n \in \mathbb{N}$.

2.2. Modules over Dedekind domains

Definition 2.2. An integral domain R is called *Dedekind*, if R is noetherian, integrally closed and every non-zero prime ideal is already maximal.

An important class of examples for Dedekind domains is given by the rings of integers of number fields (see [12, 11.88]). In the next step we characterize the finitely generated modules over Dedekind domains. Before doing this we need a further definition.

Definition 2.3. Let R be a ring. We say that an R-module M is *torsion* if, for each $m \in M$, there exists a non-zero $r \in R$ such that $rm = 0$.