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On the sign patterns of the smallest signless Laplacian eigenvector

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ABSTRACT

Let H be a connected bipartite graph, whose signless Laplacian matrix is $Q(H)$. Suppose that the bipartition of H is (S,T) and that x is the eigenvector of the smallest eigenvalue of $Q(H)$. It is well-known that x is positive and constant on S, and negative and constant on T.
The resilience of the sign pattern of x under addition of edges into the subgraph induced by either S or T is investigated and a number of cases in which the sign pattern of x persists are described.

1. Introduction

Let G be a graph with adjacency matrix $A(G)$ and let $D(G)$ be the diagonal matrix of the vertex degrees of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. The matrix $Q(G)$ has been largely subject to benign neglect up to very recent times, but it has received a lot of attention lately – some of the results of which are summarized in the surveys [5–8].

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It is well-known (cf. [5, pp. 157-158]) that 0 is an eigenvalue of $Q(G)$ with multiplicity equal to the number of bipartite connected components of G. We shall denote the smallest eigenvalue of $Q(G)$ by $\mu(G)$.

In this note we consider the relationship between bipartiteness and the signless Laplacian from a slightly different angle, studying the sign pattern of an eigenvector that corresponds to $\mu(G)$. Note also that we will restrict our attention to *real* eigenvectors.

A few words about notation: if $V(G)$ is labelled as $\{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$, then for any nonempty subset $S \subseteq V(G)$ we shall mean by $x(S)$ the vector in $\mathbb{R}^{|S|}$ formed by deleting from x all entries not corresponding to elements of S. The all-ones vector of length n will be denoted by $\mathbf{1}_n$ or just $\mathbf{1}$ if the length is clear from the context. We write $v > 0$ to indicate that all the entries of a vector v are strictly positive.

The following fact is well-known:

Proposition 1.1. *Let H be a connected bipartite graph with bipartition (S, T). For any eigenvector x corresponding to $\mu(H)$ there is a nonzero number c so that*

$$x(S) = c\mathbf{1}_{|S|}, \qquad x(T) = -c\mathbf{1}_{|T|}.$$

The main result of the paper [32] by Roth can be seen as an interesting generalization of Proposition 1.1:

Proposition 1.2. *Let H be a connected bipartite graph with bipartition (S, T). Let D be any diagonal matrix and let x be an eigenvector corresponding to the smallest eigenvalue of $Q(H) + D$. Then*

$$x(S) > 0, \qquad x(T) < 0, \tag{1}$$

or vice versa.

We are interested in generalizing Proposition 1.1, showing that (1) continues to hold even when edges are added on one of the sides of H. We keep $D = 0$, however. Let us make the following definition:

Definition 1.3. Let H be a connected graph and let $S \subseteq V(H)$ be a maximal independent set. We say that H is S-*Roth* if for every eigenvector x corresponding to $\mu(H)$ we have that

$$x(S) > 0, \qquad x\big(V(H) - S\big) < 0,$$

or vice versa.

The assumption that S is a maximal independent set is made in order to rule out the uninteresting case when H is bipartite and S is a proper subset of one of the partite sets associated with H. In that case there is a smallest eigenvector of $Q(H)$ that is positive on S but has mixed signs on the complement of S.

Remark 1.4. Notice that if H is S-Roth, then $\mu(H)$ must be a simple eigenvalue.

Proposition 1.1 can now be stated as:

Proposition 1.5. *Let H be a connected bipartite graph with bipartition (S, T). Then H is S-Roth.*