Linear Algebra and its Applications 443 (2014) 191–203

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Representations for the group inverse of anti-triangular block operator matrices

Qingxiang Xu [a,b,*,1], Chuanning Song [a,1], Lili He [a,b]

[a] Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China
[b] School of Science, Shanghai Institute of Technology, Shanghai, 201418, PR China

A R T I C L E I N F O

Article history:
Received 13 August 2013
Accepted 13 November 2013
Available online 28 November 2013
Submitted by Y. Wei

MSC:
15A09
47A05

Keywords:
Moore–Penrose inverse
Drazin inverse
Group inverse
Anti-triangular block matrix

A B S T R A C T

For any Hilbert spaces H_1 and H_2, let $\mathbb{B}(H_1, H_2)$ be the set of bounded linear operators from H_1 to H_2. In this paper, necessary and sufficient conditions are given under which an anti-triangular block operator matrix $E = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ is group invertible, where $A \in \mathbb{B}(H_1, H_1), B \in \mathbb{B}(H_2, H_1)$ and $C \in \mathbb{B}(H_1, H_2)$. In the case that E is group invertible, a new formula for the group inverse of E is derived under the only restriction that certain associated operators have closed ranges. This gives especially a new characterization of the group inverse of an anti-triangular block matrix without restrictions on its individual block matrices.

1. Introduction

Throughout this paper $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{C}^{m\times n}$ denote the sets of positive integers, complex numbers and m by n complex matrices, respectively. The notation v^T is reserved to denote the transpose of a vector v. For any (complex) Hilbert spaces H_1 and H_2, let $\mathbb{B}(H_1, H_2)$ be the set of bounded linear operators from H_1 to H_2. If $H_1 = H_2$, then $\mathbb{B}(H_1, H_1)$ is simplified to $\mathbb{B}(H_1)$. The notations of "$\oplus$" and "$\dot{+}$" are used in this paper with different meanings. For any Hilbert spaces H_1 and H_2, we let

* Corresponding author at: Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China.
E-mail addresses: qingxiang_xu@126.com, qxxu@shnu.edu.cn (Q. Xu), songning1962@163.com, songning@shnu.edu.cn (C. Song), helili@126.com, helili@sit.edu.cn (L. He).

1 Supported by the National Natural Science Foundation of China under grant 11171222, and the Innovation Program of Shanghai Municipal Education Commission under grant 12ZZ129.

http://dx.doi.org/10.1016/j.laa.2013.11.018

$$H_1 \oplus H_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, x_i \in H_i,\ i = 1,2 \right\}.$$

If both H_1 and H_2 are closed subspaces of a Hilbert space H such that $H_1 \cap H_2 = \{0\}$, then we define $H_1 \dotplus H_2 = \{x_1 + x_2 \mid x_i \in H_i,\ i = 1,2\}$. For any $A \in \mathbb{B}(H_1, H_2)$, we write $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* for its range, null space and conjugate operator, respectively. When $H_1 = H_2$, we write A^0 for the identity operator I on H_1. By a projection, we mean an idempotent and hermitian operator on a Hilbert space. Throughout the rest of this section, H, H_1 and H_2 are Hilbert spaces.

Definition 1.1. The *Moore–Penrose inverse* $A^\dagger$ of $A \in \mathbb{B}(H_1, H_2)$ is an element X of $\mathbb{B}(H_2, H_1)$ which satisfies

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX \quad \text{and} \quad (XA)^* = XA.$$

It is known that $A^\dagger$ exists if and only if A has a closed range, in which case $A^\dagger$ exists uniquely [27, Sec. 9.1], [31, Propositions 3.1.2 and 3.1.5]. By definition $(A^\dagger)^* = (A^*)^\dagger$, and both $AA^\dagger$ and $A^\dagger A$ are projections whose ranges are equal to $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively.

Definition 1.2. An operator $A \in \mathbb{B}(H)$ is said to be *Drazin invertible* (with finite index) if there exist $X \in \mathbb{B}(H)$ and $k \in \{0\} \cup \mathbb{N}$, such that

$$AX = XA, \qquad XAX = X \quad \text{and} \quad A^k = A^{k+1}X. \tag{1.1}$$

In this case, the operator X, called the *Drazin inverse* of A and written $X = A^D$, exists uniquely [27, Theorem 2.1.2]. The *Drazin index* of A, written $\operatorname{ind}(A)$, is the smallest number $k \in \{0\} \cup \mathbb{N}$ such that (1.1) holds. By definition, $\operatorname{ind}(A) = 0$ if and only if A is invertible. If $\operatorname{ind}(A) \leqslant 1$, then A is referred to be *group invertible*. In this case, the Drazin inverse of A is denoted by $A^\#$.

Remark 1.1. It is known [31, Theorem 4.1.9] that $A \in \mathbb{B}(H)$ is Drazin invertible with $\operatorname{ind}(A) \leqslant r$ if and only if

$$\mathcal{R}\left(A^r\right) = \mathcal{R}\left(A^{r+p}\right) \quad \text{and} \quad \mathcal{N}\left(A^r\right) = \mathcal{N}\left(A^{r+p}\right) \quad \text{for some } p \in \mathbb{N}.$$

In this case, H can be decomposed as $H = \mathcal{R}(A^r) \dotplus \mathcal{N}(A^r)$ such that

$$\mathcal{R}\left(A^r\right) = \mathcal{R}\left(A^D\right) \quad \text{and} \quad \mathcal{N}\left(A^r\right) = \mathcal{N}\left(A^D\right). \tag{1.2}$$

The Drazin inverse, including the group inverse as a special case, has applications in differential-algebraic equations [1,3,6–8,18], Markov chains [21], linear stationary iterative processes [22], Leslie models of population [15], weighted trees [16,17] and so on. Motivated by various applications of the Drazin inverse, much progress has been made concerning the representation for the Drazin inverse of a block matrix, or more generally, of a block operator matrix. Specifically, let $E = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$ be a block matrix such that both A and E are square. Campbell [6,7] showed that the Drazin inverse E^D of such an anti-triangular square matrix E, can be used to express the solutions of certain second-order singular linear differential equations. In recent years, much attention has been paid in representations for above E^D under various additional restrictions imposed on the individual blocks A, B and C [2–5, 9,12–14,19,20,25,29,32]. It has long been an open problem [7] to derive a formula for E^D in terms of the blocks of the partition without restrictions on the individual blocks. Although this problem still remains open up to now for the general Drazin inverse, it has been solved in the special case of the group inverse over a ring under a mild restriction that certain elements are von Neumann regular [10,24].

In this paper, we study the group inverse in the setting of bounded linear operators on Hilbert spaces. Let E be an anti-triangular operator matrix defined by

$$E = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathbb{B}(H_1 \oplus H_2), \tag{1.3}$$