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Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues ☆

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ABSTRACT

In this paper, we show that the largest signless Laplacian H-eigenvalue of a connected k-uniform hypergraph G, where $k \geqslant 3$, reaches its upper bound $2\Delta(G)$, where $\Delta(G)$ is the largest degree of G, if and only if G is regular. Thus the largest Laplacian H-eigenvalue of G, reaches the same upper bound, if and only if G is regular and odd-bipartite. We show that an s-cycle G, as a k-uniform hypergraph, where $1 \leqslant s \leqslant k-1$, is regular if and only if there is a positive integer q such that $k = q(k-s)$. We show that an even-uniform s-path and an even-uniform non-regular s-cycle are always odd-bipartite. We prove that a regular s-cycle G with $k = q(k-s)$ is odd-bipartite if and only if m is a multiple of 2^{t_0}, where m is the number of edges in G, and $q = 2^{t_0}(2l_0+1)$ for some integers t_0 and l_0. We identify the value of the largest signless Laplacian H-eigenvalue of an s-cycle G in all possible cases. When G is odd-bipartite, this is also its largest Laplacian H-eigenvalue. We introduce supervertices for hypergraphs, and show the components of a Laplacian H-eigenvector of an odd-uniform hypergraph are equal if such components correspond vertices in the same supervertex, and the corresponding Laplacian H-eigenvalue is not equal to the degree of the supervertex. Using this property, we show that the largest Laplacian H-eigenvalue of an odd-uniform generalized loose s-cycle G is equal to $\Delta(G) = 2$. We also show that the largest Laplacian H-eigenvalue of a k-uniform tight s-cycle G is not less than $\Delta(G)+1$, if the number of vertices is even and $k = 4l+3$ for some nonnegative integer l.

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1. Introduction

Let $k \geqslant 2$ and $n \geqslant k$. A k-uniform hypergraph $G = (V, E)$ has vertex set V, which is labeled as $[n] = \{1, \dots, n\}$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. If $k = 2$, we have an ordinary graph.

The largest Laplacian eigenvalue of a graph plays an important role in spectral graph theory [1,17]. A natural definition for the Laplacian and signless Laplacian tensors of a k-uniform hypergraph G, where $k \geqslant 3$, was introduced in [16]. It was shown that the largest Laplacian H-eigenvalue of G is always less than or equal to the largest signless Laplacian H-eigenvalue of G, while the latter is always less than or equal to 2Δ, where Δ is the largest degree of G. In [7], the odd-bipartite hypergraph was introduced. In [9], it was proved that the largest Laplacian H-eigenvalue of a connected k-uniform hypergraph G is equal to its largest signless Laplacian H-eigenvalue if and only if G is odd-bipartite. This result extended the classical result in spectral graph theory [1,17].

In this paper, we show that the largest signless Laplacian H-eigenvalue of a connected k-uniform hypergraph G, where $k \geqslant 3$, reaches its upper bound 2Δ, if and only if G is regular. Thus, the largest Laplacian H-eigenvalue of G reaches the same upper bound, if and only if G is regular and odd-bipartite.

We then turn our attention to s-paths and s-cycles.

Researchers in hypergraph theory have studied loose cycles, loose paths, tight cycles and tight paths extensively [3–6,10–14].

Let $G = (V, E)$ be a k-uniform hypergraph. Suppose $1 \leqslant s \leqslant k-1$. According to [14], if $V = \{i\colon\ i \in [s + m(k-s)]\}$ such that $\{1 + j(k-s), \dots, s + (j+1)(k-s)\}$ is an edge of G for $j = 0, \dots, m-1$, then G is called an s-path. In [13], G is called a *loose path* if $s = 1$, and a *tight path* if $s = k-1$. In [14], G is also called a loose path for $2 \leqslant s \leqslant \frac{k}{2}$ and a tight path for $\frac{k}{2} < s \leqslant k-2$. To avoid confusion, in these two cases, as in [8], we call G a *generalized loose path* and a *generalized tight path* respectively. According to [14], if $V = \{i\colon\ i \in [m(k-s)]\}$ such that $\{1 + j(k-s), \dots, s + (j+1)(k-s)\}$ is an edge of G for $j = 0, \dots, m-1$, where vertices $m(k-s) + j \equiv j$ for any j, then G is called an s-cycle. According to [3–6,10–13], if $s = 1$, G is called a *loose cycle*, if $s = k-1$, G is called a *tight cycle*. We call G a *generalized loose cycle* for $2 \leqslant s \leqslant \frac{k}{2}$, and a *generalized tight cycle* for $\frac{k}{2} < s \leqslant k-2$. For an s-cycle, in this paper, we assume that $n \geqslant 2k - s$. In this way, each pair of consecutive edges in the s-cycle will have exactly s common vertices. In the next section, we will discuss this in details.

We show in this paper that an s-cycle G, as a k-uniform hypergraph, where $1 \leqslant s \leqslant k-1$, is regular if and only if k is a multiple of $k - s$.

The Laplacian H-eigenvalues of loose paths and loose cycles were studied in [8,9]. In [8], power hypergraphs and cored hypergraphs were introduced. Loose paths and loose cycles are power hypergraphs. Power hypergraphs are cored hypergraphs. Even-uniform cored hypergraphs are odd-bipartite. As cycles are symmetric, their largest signless Laplacian H-eigenvalues can be identified directly. Thus, the largest Laplacian H-eigenvalues of odd-bipartite cycles can be identified directly. In [9], the largest Laplacian H-eigenvalue of an even-uniform loose cycle was identified directly.

According to [16], the largest Laplacian H-eigenvalue of k-uniform hypergraph is always greater than or equal to the largest degree of that k-uniform hypergraph. By [9], when k is even, equality cannot hold, but when k is odd, equality may hold in certain cases. It was proved in [8] that equality holds for odd-uniform loose paths and loose cycles.

It was observed in [8] that if $2 \leqslant s < \frac{k}{2}$, then an s-path or an s-cycle is a cored hypergraph, but not a power hypergraph in general.

These results raised several questions. First, if k is even and $\frac{k}{2} \leqslant s \leqslant k-1$, are some s-paths and s-cycles still odd-bipartite, though they are not cored hypergraphs? Second, can we identify the largest Laplacian H-eigenvalues of even-uniform odd-bipartite s-cycles directly? Third, when k is odd and $2 \leqslant s \leqslant k-1$, are the largest H-eigenvalues of s-paths and s-cycles equal to the corresponding largest degrees? We will study these questions in this paper.

We give some basic definitions in the next section.

In Section 3, we prove the result about regular uniform hypergraphs mentioned before, and identify regular s-cycles.