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Equivalence and normal forms of bilinear forms

Fernando Szechtman [1]

Department of Mathematics and Statistics, University of Regina, Canada

A R T I C L E I N F O

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A B S T R A C T

We present an alternative account of the problem of classifying and finding normal forms for arbitrary bilinear forms. Beginning from basic results developed by Riehm, our solution to this problem hinges on the classification of indecomposable forms and in how uniquely they fit together to produce all other forms. We emphasize the use of split forms, i.e., those bilinear forms such that the minimal polynomial of the asymmetry of their non-degenerate part splits over ground field, rather than restricting the field to be algebraically closed. In order to obtain the most explicit results, without resorting to the classification of hermitian, symmetric and quadratic forms, we merely require that the underlying field be quadratically closed.

1. Introduction

The problem of classifying arbitrary bilinear forms up to equivalence, that is, arbitrary square matrices up to congruence, has been solved through the work of Williamson [25–27], Wall [24], Riehm [17] and Gabriel [8].

Over a general field, the solution consists of reducing the classification to the case of hermitian and symmetric forms, although in characteristic 2 this reduction involves the classification of quadratic forms, as well. It would perhaps be fair to refer to this as a relative solution.

Over an algebraically closed field it is possible to solve the equivalence problem explicitly, as done by Riehm [17]. Moreover, very simple normal forms have been obtained in this case by Horn and Sergeichuk [13].

Somewhere in between lies the case of *split* forms. We define these as bilinear forms f such that the minimal polynomial of the asymmetry of the non-degenerate part of f splits over ground field (these terms are defined below). Of course, every bilinear form defined over an algebraically closed

E-mail address: fernando.szechtman@gmail.com.

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field is split, but this requirement is perhaps too demanding. All bilinear forms considered in this paper will be assumed to be split. This is similar in spirit to Jacobson's [14] decision to consider split semisimple Lie algebras in characteristic 0, rather than restricting the ground field to be algebraically closed.

In this paper, we furnish an alternative account of how the classification problem can be solved and how normal forms can be produced, in the case of split forms. The end results are explicit and the means to arrive at them are extremely simple. To achieve these goals we must sacrifice generality by requiring the ground field to be quadratically closed at certain strategic points.

Details of our strategy and prior work on the subject are discussed below. Let us start, however, by reviewing some of the known results on the classification of bilinear forms in the classical case of alternating and symmetric forms. In either case, after splitting the radical, one is reduced to consider non-degenerate forms only.

It is well-known that a finite dimensional vector space V admits a non-degenerate alternating form if and only if $\dim(V)$ is even, in which case any two such forms defined on V are equivalent.

For non-degenerate symmetric bilinear forms the classification is actually field dependent. Let $f, g : V \times V \to F$ be two such forms, where F is a field and V is a finite dimensional vector space over F. Let us write $f \sim g$ to mean that f and g are equivalent.

Suppose first that F is quadratically closed. If $\operatorname{char}(F) \neq 2$ then $f \sim g$. If $\operatorname{char}(F) = 2$ then $f \sim g$ if and only if f and g are both alternating or both non-alternating.

Over the reals, $f \sim g$ if and only if they have the same signature; over a finite field the decisive condition is the discriminant in characteristic not 2, and simply the alternating or non-alternating nature of the forms in characteristic 2; over the p-adic field $\mathbb{Q}_p$ the invariants are the discriminant and the Hasse symbol, while $f \sim g$ over $\mathbb{Q}$ if and only if $f \sim g$ over $\mathbb{R}$ and $\mathbb{Q}_p$ for every prime p. See [15] for details about the above cases, except the last which can be found in [16], which also treats the more general cases of arbitrary local and global fields.

Let us move to the case of arbitrary bilinear forms. The starting point is the following result of Gabriel [8]: Any bilinear form, say f, decomposes as the orthogonal direct sum of finitely many indecomposable degenerate forms and a non-degenerate form. Moreover, all summands in this decomposition are uniquely determined by f, up to equivalence. The uniqueness is really only up to equivalence, as shown by Djokovic and Szechtman [5] in their study of the isometry group of an arbitrary bilinear form. In this regard, see [20] and [3]. Gabriel's result effectively reduces the classification problem to the case of non-degenerate forms. In particular, Gabriel determines all indecomposable degenerate forms; there is only one for each dimension, and we will refer to it as a Gabriel block. The above formulation of Gabriel's result was given by Waterhouse [28], who also included a matrix appearance of Gabriel's blocks. Waterhouse used Gabriel's decomposition to compute the number of congruence classes in $M_n(F_q)$, following Gow [9], who had previously obtained the corresponding result for $\mathrm{GL}_n(F_q)$, perhaps unaware of [8]. In his proof, Gabriel uses the theory of Kronecker modules (better known to linear algebraists as the theory of matrix pencils) and in particular the Krull–Remak–Schmidt theorem for these modules. An elementary proof of the uniqueness part of Gabriel's result was given by Djokovic and Szechtman [4], in the more general context of sesquilinear forms over semisimple artinian rings with an involution. A simple proof of the existence of Gabriel's decomposition was furnished by Djokovic, Szechtman and Zhao [6]. An alternative new short proof is given in Section 3. We note that [6] gives an algorithm that given any $A \in M_n(F)$ produces $X \in \mathrm{GL}_n(F)$ such that $X^2 = I$ and $X'AX = A'$, the transpose of A. The existence of such X had first been obtained by Gow [10] for $A \in \mathrm{GL}_n(F)$, and then by Yip and Ballantine [29] for $A \in M_n(F)$. Again, the missing link was Gabriel's decomposition. We remark that [29] is a continuation of prior work of Yip and Ballantine [30] on the equivalence of bilinear forms. A short proof of the existence of X for any $A \in M_n(F)$ was given by Horn and Sergeichuk [11] using the study made by Sergeichuk [21] of the classification of bilinear forms. An algorithmic view of Gabriel's decomposition is given by Horn and Sergeichuk in [12].

Let us consider next the case of non-degenerate bilinear forms and, more generally, the case of a non-degenerate sesquilinear form $f : V \times V \to D$ defined over a finite dimensional vector space V over a division ring D with involution J. Then f admits a unique asymmetry $\sigma \in \mathrm{GL}(V)$, satisfying

$$f(v,u)^J = f(u, \sigma v), \quad u, v \in V,$$