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On a unified view of nullspace-type conditions for recoveries associated with general sparsity structures

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ABSTRACT

We discuss a general notion of "sparsity structure" and associated recoveries of a sparse signal from its linear image of reduced dimension possibly corrupted with noise. Our approach allows for unified treatment of (a) the "usual sparsity" and "usual ℓ_1 recovery", (b) block-sparsity with possibly overlapping blocks and associated block-ℓ_1 recovery, and (c) low-rank-oriented recovery by nuclear norm minimization. The proposed recovery routines are natural extensions of the usual ℓ_1 minimization used in Compressed Sensing. Specifically, within this framework, we present nullspace-type sufficient conditions for the recovery to be precise on sparse signals in the noiseless case. Then we derive error bounds for imperfect (nearly sparse signal, presence of observation noise, etc.) recovery under these conditions. In all of these cases, we present efficiently verifiable sufficient conditions for the validity of the associated nullspace properties.

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1. Introduction

We address the problem of recovering a *representation* $w = Bx \in E$ of an unknown signal $x \in X$ via noisy observations

$$y = Ax + \xi$$

of x. Here X, E are Euclidean spaces, $A : X \to \mathbf{R}^m$ and $B : X \to E$ are given linear *sensing* and *representation* maps, and ξ is "uncertain-but-bounded" observation error satisfying $\phi(\xi) \leqslant \epsilon$ where $\phi(\cdot)$ is a given norm on $\mathbf{R}^m$, and ϵ is a given error bound. In particular, we are interested in computationally tractable recovery procedures (estimators), e.g., a recovering routine of the form

$$y \mapsto \hat{x}(y) \in \operatorname*{Argmin}_{u \in X} \bigl\{ \|Bu\|: \phi(Au - y) \leqslant \epsilon \bigr\} \mapsto \hat{w}(y) = B\hat{x}(y),$$

and we would like this recovery to behave well provided that Bx is sparse in some prescribed sense. In this paper, we introduce a rather general notion of *sparsity structure* in the representation space E which, under some structural restriction on the norm $\|\cdot\|$, allows us to point out "nullspace-type" conditions for the recovery to be precise provided that Bx is "s-sparse" with respect to our structure. It also allows for explicit error bounds for "imperfect recovery" (noisy observations, near s-sparsity instead of the exact one, etc.). To streamline Introduction, we assume here (but not in the main body of the paper!) that $E = X$ and B is the identity mapping, so that what is sparse "in some prescribed sense" and what we want to recover is the signal x itself.

The motivation behind our notion of sparsity structure is to present a simple unified general framework which allows, for instance, a simple treatment of three important particular cases:

- recovering s-sparse signals via ℓ_1 minimization,[2]
- recovering s-block-sparse signals via block-ℓ_1 minimization, and
- recovering matrices of low rank via nuclear norm minimization.

Within the past decade, each one of the above cases has individually received enormous and still growing attention. The theory for sparse and block-sparse recovery, also referred to as Compressed Sensing, goes back to [6,10–12,28]. One of its principal results states that if x is s-sparse and the linear sensing map A possesses a certain well-defined property referred to as *nullspace property*, then x can be reconstructed from the observation $y = Ax$ via ℓ_1 minimization. It is also known that under the (properly quantified) nullspace property, ℓ_1 recovery admits explicit error bounds when x is just approximately sparse and/or the observations are corrupted by noise; these error bounds are just linear in naturally quantified "sparsity violation" and the level of observation noise. Furthermore, it was shown that a large class of random matrices satisfies a specific sufficient condition, the *restricted isometry property* (RIP), for the nullspace property (see [10]).

A problem closely related to (block-)sparse recovery is that of *low-rank recovery*, which has many applications in a diverse set of fields (for further details, see [7,23,24,26] and references therein). Nuclear norm minimization is used as a computationally efficient tool for handling this difficult problem. The first formal performance guarantee for nuclear norm minimization is given in [23] under the matrix RIP condition, a natural extension of the RIP condition used in sparse recovery. RIP appears to be the most commonly used sufficient condition on the sensing map A for ensuring the validity of recovery via (block-)ℓ_1/nuclear norm minimization. This condition is as follows: Given a positive integer k and real $\delta \in (0, 1)$, a linear sensing map A is said to possess RIP(k, δ) if

$$(1 - \delta)\|x\|^2 \leqslant \|Ax\|_2^2 \leqslant (1 + \delta)\|x\|^2$$

holds for all vectors x that have at most k nonzero entries/blocks in the case of usual/block-sparse recovery, or all matrices x of rank at most k in the case of low-rank matrix recovery; here $\|\cdot\|$ is the

[2] Here *s-sparse* is used in the usual sense, i.e., vectors with at most s nonzero entries.