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A note on the multiplicities of graph eigenvalues

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ABSTRACT

Let G be a graph with vertex set $\{1,\dots,n\}$, and let H be the graph obtained by attaching one pendant path of length k_i at vertex i $(i=1,\dots,r, 1\leqslant r\leqslant n)$. For a real symmetric matrix A whose graph is H, let $m_A(\mu)$ denote the multiplicity of an eigenvalue μ of A. From a result in da Fonseca (2005) [7], we know that $m_A(\mu)\leqslant n$. In this note, we characterize the case $m_A(\mu)=n$. We also give two upper bounds on eigenvalue multiplicity of trees and unicyclic graphs, which are generalizations of some results in Rowlinson (2010) [10].

1. Introduction

All graphs in this paper are simple undirected graphs. For a graph G with vertex set $V(G)=\{1,\dots,n\}$, the adjacency matrix of G is the matrix $A=(a_{ij})$, where $a_{ij}=1$ if there is an edge between vertices i and j, and 0 otherwise. The eigenvalues of A are called eigenvalues of G. If μ is an eigenvalue of G of multiplicity k, then a *star set* for μ in G is a subset X of $V(G)$ such that $|X|=k$ and the induced subgraph $G-X$ does not have μ as an eigenvalue. The induced subgraph $G-X$ is called a *star complement* for μ in G. It is known that star sets and star complements exist for any eigenvalue of any graph (see [5]). The research on star complements originated independently in papers by Ellingham [6] and Rowlinson [9]. The theory of star complement has been used to study graphs with least eigenvalue -2 [2,4], structure properties of strongly regular graphs [13,14] and the multiplicities of graph eigenvalues [1,10–12].

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Let $A=(a_{ij})_{n\times n}$ be a real symmetric matrix. In [7,8], the graph of A, denoted by $G(A)$, is defined as follows: the vertex set is $\{1,\dots,n\}$ and i,j are adjacent if and only if $a_{ij}\neq 0$. If A is a $(0,1)$-matrix with zero diagonal entries, then A is the adjacency matrix of $G(A)$. For a graph G, let $\mathcal{S}(G)$ denote the set of all real symmetric matrices sharing a common graph G (see [7,8]).

A vertex of degree 1 is called a *pendant vertex*. We say that the path $u_0u_1\cdots u_s$ is a *pendant path* of length s if $d(u_0)=1, d(u_1)=\cdots=d(u_{s-1})=2$, where $d(u_i)$ is the degree of vertex u_i. Moreover, we say that the pendant path $u_0u_1\cdots u_s$ is *proper* if $d(u_0)=1, d(u_1)=\cdots=d(u_{s-1})=2, d(u_s)\neq 1$. Let C_n and P_n denote the cycle and the path of order n, respectively.

Let G be a graph with vertex set $\{1,\dots,n\}$, and let H be the graph obtained by attaching one pendant path of length k_i at vertex i $(i=1,\dots,r, 1\leqslant r\leqslant n)$. For $A\in\mathcal{S}(H)$, let $m_A(\mu)$ denote the multiplicity of an eigenvalue μ of A. From [7, Theorem 3.1] and [8, Corollary 2.3], we know that $m_A(\mu)\leqslant n$. In this note, we characterize the case $m_A(\mu)=n$. We also obtain two upper bounds on eigenvalue multiplicity of trees and unicyclic graphs, which are generalizations of some results in [10].

2. Preliminaries

In order to obtain our main results, we give the following lemmas.

Lemma 2.1. *(See [5].) Let X be a set of k vertices in graph G, and suppose that G has adjacency matrix $\begin{pmatrix} A & B^\top \\ B & C\end{pmatrix}$, where A is the adjacency matrix of the subgraph induced by X. Then X is a star set for an eigenvalue μ of G if and only if μ is not an eigenvalue of C and*

$$\mu I-A=B^\top(\mu I-C)^{-1}B.$$

Lemma 2.2. *(See [5].) Let μ be an eigenvalue of a connected graph G, and let K be a connected induced subgraph of G not having μ as an eigenvalue. Then G has a connected star complement for μ containing K.*

Lemma 2.3. *(See [10].) If u and v are adjacent vertices in a star set for G, then the edge uv is not a bridge of G.*

Lemma 2.4. *Let $f_0(\mu)=1$, $f_1(\mu)=\mu$, $f_n(\mu)=\mu f_{n-1}(\mu)-f_{n-2}(\mu)$ $(n=2,3,\dots)$. Then $f_n(\mu)=0$ $(n\geqslant 1)$ if and only if $\mu\in\{2\cos\frac{k\pi}{n+1}; k=1,\dots,n\}$.*

Proof. The Chebyshev polynomials of the second kind are defined by the recurrence relation $U_0(x)=1$, $U_1(x)=2x$, $U_n(x)=2xU_{n-1}(x)-U_{n-2}(x)$. The roots of $U_n(x)$ are $\cos\frac{k\pi}{n+1}$ $(k=1,\dots,n)$. Since $f_n(\mu)=U_n(\mu/2)$, we know that $f_n(\mu)=0$ if and only if $\mu\in\{2\cos\frac{k\pi}{n+1}; k=1,\dots,n\}$. □

Lemma 2.5. *(See [7].) If $A\in\mathcal{S}(P_n)$, then A has n distinct real eigenvalues.*

3. Main results

Let $A=(a_{ij})_{n\times n}$ be a real symmetric matrix. For any eigenvalue μ of A, the *eigenspace* of μ is defined as $\mathcal{E}(\mu)=\{x\in\mathbb{R}^n: Ax=\mu x\}$. Let $\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n\}$ denote the standard orthonormal basis, and let E_μ denote the matrix which represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$ of A with respect to $\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n\}$. There always exists $X\subseteq\{1,\dots,n\}$ such that vectors $E_\mu\mathbf{e}_i$ $(i\in X)$ form a basis for $\mathcal{E}(\mu)$, such a set X is called a *star basis* for eigenvalue μ of A (see [3]). Clearly $|X|=\dim\mathcal{E}(\mu)$ is the multiplicity of μ. If A is a $(0,1)$-matrix with zero diagonal entries, i.e., A is the adjacency matrix of a graph G, then X is a star basis for μ if and only if X is a star set for μ in G (see [5]). Since E_μ is a polynomial function of A, we have $\mu E_\mu\mathbf{e}_i=AE_\mu\mathbf{e}_i=E_\mu A\mathbf{e}_i=\sum_{j=1}^n a_{ji}E_\mu\mathbf{e}_j$. If $A\in\mathcal{S}(G)$, then

$$(\mu-a_{ii})E_\mu\mathbf{e}_i=\sum_{j\sim i}a_{ji}E_\mu\mathbf{e}_j, \tag{1}$$

where $j\sim i$ means that i and j are adjacent in graph G.