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An upper bound for the energy of radial digraphs

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ABSTRACT

Let D be a digraph with n vertices, a arcs, c_2 closed walks of length 2 and spectral radius $\rho(D)$. Recently Ayyaswamy, Balachandran and Gutman (2011) [1] proved that when

$$\rho(D) \geqslant \frac{a+c_2}{2n} \geqslant 1$$

it is possible to construct an upper bound for the energy of digraphs, which improves the McClelland inequality for the energy of strongly connected digraphs given in Rada (2009) [16]. It is our interest in this paper to show that for general digraphs the inequality

$$\rho(D) \geqslant \frac{a+c_2}{2n}$$

does not hold. However, we introduce the class of radial digraphs, that satisfy the spectral radius condition above, and for this class of digraphs we improve the bound for the energy given in Ayyaswamy et al. (2011) [1].

1. Introduction

A digraph D consists of a non-empty set of vertices $\mathcal{V}$ and a finite set $\mathcal{A}$ of ordered pairs of vertices called arcs. We only consider digraphs with no loops and no multiple arcs. Two vertices are adjacent if they are connected by an arc. If there is an arc from the vertex u to the vertex v, we indicate this by writing uv.

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The adjacency matrix of a digraph D with vertices $\{v_1, \ldots, v_n\}$ is the $n \times n$ matrix A defined as follows:

$$[A]_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

where $[A]_{ij}$ is the ij-entry of the matrix A. The characteristic polynomial of the digraph D is defined as the characteristic polynomial of A, and it is denoted by Φ_D. The eigenvalues of the digraph D are the eigenvalues of the adjacency matrix A, in other words, the roots of the polynomial Φ_D. The spectral radius of a digraph D is denoted by $\rho = \rho(D)$ and defined as

$$\rho = \max_{1 \leqslant i \leqslant n} \{|z_i|\}$$

where $z_1, \ldots, z_n$ are the (possibly complex) eigenvalues of D. We refer the reader to the books [5,10] for further terminology and spectral properties of the adjacency matrix.

A digraph D is symmetric if $uv \in \mathcal{A}$ then $vu \in \mathcal{A}$, for all $u, v \in \mathcal{V}$. A one-to-one correspondence between simple graphs and symmetric digraphs is given by $G \rightsquigarrow \overleftrightarrow{G}$, where $\overleftrightarrow{G}$ has the same vertex set as the graph G, and each edge uv of G is replaced by a pair of symmetric arcs uv and vu. Under this correspondence, a graph G can be identified with a symmetric digraph.

The energy of a graph G was defined by Gutman [7] as

$$E(G) = \sum_{k=1}^{n} |\lambda_k|$$

where n is the number of vertices of G and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G. It is extensively studied in chemistry and mathematics, for surveys of the properties of graph energy we refer to [8,9, 14]. Recently, this concept was generalized to digraphs [15] as

$$E(D) = \sum_{k=1}^{n} \left|\mathrm{Re}(z_k)\right|$$

where $z_1, \ldots, z_n$ are the eigenvalues of D. This definition was motivated by Coulson's integral formula for digraphs [15, Theorem 4.1]

$$E(D) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{ix\Phi'_D(ix)}{\Phi_D(ix)} \right] dx$$

For further study of the mathematical properties of the energy of digraphs we refer to [1,2,6,15–18].

Let D be a digraph with n vertices, a arcs and c_2 closed walks of length 2. Recently, Ayyaswamy, Balachandran and Gutman [1] proved that if

$$\rho(D) \geqslant \frac{a + c_2}{2n} \geqslant 1 \tag{1}$$

then the inequality

$$E(D) \leqslant \frac{a + c_2}{2n} + \sqrt{(n-1)\left[\frac{a + c_2}{2} - \left(\frac{a + c_2}{2n} \right)^2 \right]} \tag{2}$$

holds. Moreover, they showed that this bound is an improvement of the McClelland inequality for the energy of strongly connected digraphs given in [16]. They also state that the problem of characterizing digraphs which satisfy inequality (1) is an open problem. It is our interest in this paper to show that for general digraphs the inequality

$$\rho(D) \geqslant \frac{a + c_2}{2n} \tag{3}$$

does not hold. However, we introduce a large class of digraphs, which we call radial digraphs, that satisfy condition (3), and for this class of digraphs, we improve the bound for the energy given in (2).