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Star complements and connectivity in finite graphs

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ABSTRACT

Let G be a finite graph with H as a star complement for an eigenvalue other than 0 or -1. Let $\kappa(G)$, $\delta(G)$ denote respectively the vertex-connectivity and minimum degree of G. We prove that $\kappa(G)$ is controlled by $\delta(G)$ and $\kappa(H)$. In particular, for each $k \in \mathbb{N}$ there exists a smallest non-negative integer $f(k)$ such that $\kappa(G) \geqslant k$ whenever $\kappa(H) \geqslant k$ and $\delta(G) \geqslant f(k)$. We show that $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$ and $f(5) = 7$.

1. Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. We use the notation of [7], where the fundamental properties of star sets and star complements are established in Chapter 5.

It is well known that if $\mu \neq -1$ or 0 and $n > 4$ then $|X| \leqslant \binom{n-k}{2}$ [1]; in particular, there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1. Certain graphs can be characterised by a star complement: for surveys, see [9] and [11]. More generally, it is of interest to investigate properties of H that are reflected in G: connectedness is one

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such property, as noted in [8, Section 2]. Here we discuss k-connectedness for $k > 1$. In Section 2 we show that for each $k \in \mathbb{N}$ there exists a non-negative integer $F(k)$ with the following property: if $\mu \notin \{-1, 0\}$, H is k-connected and G has least degree $\delta(G) \geqslant F(k)$ then G is k-connected. It is straightforward to show that if $f(k)$ is the smallest non-negative integer with this property, then $f(1) = 0$ and $f(2) = 2$. In Sections 3 and 4 we show that $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $8 \leqslant f(6) \leqslant 20$.

We take $V(G) = \{1, \dots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S, and $\Delta_S(u)$ for the S-neighbourhood $\{v \in S\colon v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. Let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

We shall require the following properties of star sets and star complements; the first follows from [7, Proposition 5.1.1].

Lemma 1.1. *The subset S of $V(G)$ lies in a star set for μ if and only if the vectors $P\mathbf{e}_i$ $(i \in S)$ are linearly independent.*

Since P is a polynomial in A [7, Eq. 1.5] we have $\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i$ $(i = 1, \dots, n)$, whence:

Lemma 1.2. $\mu P\mathbf{e}_i = \sum_{j \sim i} P\mathbf{e}_j$ $(i = 1, \dots, n)$.

As a consequence of Lemmas 1.1 and 1.2 we have:

Lemma 1.3. *(See [7, Proposition 5.1.4].) Let X be a star set for μ in G, and let $H = G - X$. If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set in G, that is, the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.*

By interlacing [7, Corollary 1.3.12] we have:

Lemma 1.4. *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

The next result strengthens [1, Theorem 2.3], which says that if G has H as a star complement of order t, for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$.

Proposition 1.5. *Let G be a graph with X as a star set for μ, and let $H = G - X$. Let $s = |\bigcup_{i \in X} \Delta_H(i)|$.*

(i) *If $|X| > s$ then G_X has μ as an eigenvalue of multiplicity at least $|X| - s$.*
(ii) *If $\mu \notin \{-1, 0\}$ then $|X| \leqslant \binom{s+1}{2}$.*

Proof. Since $s \leqslant \binom{s+1}{2}$, the second assertion is immediate when $|X| \leqslant s$. Accordingly we assume throughout the proof that $|X| > s$. We show first that μ is an eigenvalue of G_X. Let $S = \bigcup_{i \in X} \Delta_H(i)$ and $X = \{1, 2, \dots, k\}$. By Lemma 1.2, the vectors $\mu P\mathbf{e}_i - \Sigma\{P\mathbf{e}_j\colon\ j \in \Delta_X(i)\}$ $(i \in X)$ lie in the subspace $\langle P\mathbf{e}_h\colon\ h \in S\rangle$, and so there exist $\alpha_1, \alpha_2, \dots, \alpha_k$, not all zero, such that

$$\sum_{i=1}^{k} \alpha_i \bigl(\mu P\mathbf{e}_i - \Sigma\bigl\{P\mathbf{e}_j\colon\ j \in \Delta_X(i)\bigr\}\bigr) = \mathbf{0}.$$

Since the vectors $P\mathbf{e}_i$ $(i \in X)$ are linearly independent, it follows that $(\mu I - A_X)\mathbf{a} = \mathbf{0}$, where $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_k)^\top$ and A_X is the adjacency matrix of G_X.

Let Y be a star set for μ in G_X and consider the graph $G - Y$. If $|X| - |Y| > s$ then the above argument shows that μ is an eigenvalue of $G_{X \setminus Y}$. This is a contradiction because $G_{X \setminus Y}$ is a star complement for μ in G_X. Hence $|Y| \geqslant |X| - s$, and we have proved the first assertion.