Linear Algebra and its Applications 442 (2014) 99–105

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Note on the spectral radius of alternating sign matrices

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ARTICLE INFO

Article history:
Received 4 March 2013
Accepted 4 August 2013
Available online 22 August 2013
Submitted by P. Semrl

MSC:
05B20
05C22
05C50
15B36

Keywords:
Alternating sign matrix
ASM
Spectral radius

ABSTRACT

We show that the $n \times n$ so-called diamond alternating sign matrix D_n is the unique $n \times n$ alternating sign matrix with maximum spectral radius ρ_n, and that $\lim_{n\to\infty} \frac{\rho_n}{n} = \frac{2}{\pi}$.

An *alternating sign matrix*, abbreviated to ASM, is an $n \times n$ $(0, +1, -1)$-matrix such that, ignoring 0s, the $+1$s and -1s alternate in each row and column beginning and ending with a $+1$. In particular, all row and column sums of an ASM equal 1. For example, the matrix

$$\begin{bmatrix} 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix} \tag{1}$$

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.08.004

is an ASM. Every permutation matrix is an ASM, and so ASMs can be thought of as generalizations of permutation matrices. A row and column permutation of an ASM, even a simultaneous row and column permutation, need not result in an ASM. For instance, the matrix obtained by interchanging rows 1 and 2 and columns 1 and 2 of the ASM in (1) is not an ASM. We refer the reader to [2] for the history of ASMs and to [1] for a recent combinatorial investigation of ASMs.

Let $\mathcal{A}_n$ denote the set of all $n \times n$ ASMs. Let $\rho(X)$ denote the *spectral radius* of the $n \times n$ matrix X, that is, the maximum absolute value of an eigenvalue of X. Since all row and column sums of an ASM equal 1, it follows that 1 is an eigenvalue of every ASM with corresponding eigenvector equal to a vector of all 1s. Thus $\rho(A) \geqslant 1$ for every ASM; equality occurs for ASMs that are permutation matrices but may occur for other ASMs as well. For instance, let

$$A = \begin{bmatrix} 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is easily computed to be

$$(\lambda - 1)^2(\lambda + 1)\left(\lambda^2 - \lambda + 1\right)$$

and hence A has eigenvalues

$$1, 1, -1, \frac{1 \pm \sqrt{-3}}{2}$$

all of which have absolute value equal to 1. Therefore $\rho(A) = 1$. This also follows from the fact that $A^6 = I_6$ (in fact, all powers of A are $(0, +1, -1)$-matrices).

The purpose of this note is to determine the maximum spectral radius of an $n \times n$ ASM:

$$\rho_n = \left\{\max \rho(A)\colon\ A \in \mathcal{A}_n\right\}.$$

Although every ASM different from a permutation matrix has at least one negative entry, we nonetheless are able to make use of the well-known Perron–Frobenius theory of nonnegative matrices, abbreviated to (PF), (see e.g. Chapter 8 of [3]). We also use the fact that for a square matrix B, $\rho(B^2) = \rho(B)^2$, and we make use of the following theorem (actually the technique of its proof) of Schwarz [4], which is a consequence of this theory.

Theorem 1. *Let $c_1, c_2, \ldots, c_{n^2}$ be n^2 nonnegative numbers. Let $\mathcal{C}_n$ be the set of all $n \times n$ matrices whose entries are these n^2 numbers in some order. Then the maximum spectral radius of a matrix in $\mathcal{C}_n$ is attained by a matrix $A = [a_{ij}]$ in $\mathcal{C}_n$ for which the entries in each row and in each column are nonincreasing*:

$$a_{i1} \geqslant a_{i2} \geqslant \cdots \geqslant a_{in} \quad (1 \leqslant i \leqslant n), \quad \textit{and} \quad a_{1j} \geqslant a_{2j} \geqslant \cdots \geqslant a_{nj} \quad (1 \leqslant j \leqslant n).$$

In replacing in Theorem 1 the maximum spectral radius with minimum spectral radius and nonincreasing with nondecreasing, an equally valid theorem results.

If X is an $n \times n$ real matrix, then we denote by $|X|$ the $n \times n$ nonnegative matrix obtained from X by replacing each entry with its absolute value. If A is an $n \times n$ ASM, then $|A|$ is a $(0, 1)$-matrix. Counting the number of 1s in each row of $|A|$ we get the *row sum vector* $R_n = (r_1, r_2, \ldots, r_n)$ of $|A|$ where each r_i is necessarily odd. The alternating property of rows of an ASM implies that the $R_n \leqslant T_n$ entrywise where $T_n = (1, 3, 5, \ldots, 5, 3, 1)$. Similarly, we have a column sum vector S_n of $|A|$ with all odd components satisfying $S_n \leqslant T_n$ entrywise. The *diamond ASM* D_n is the $n \times n$ ASM for which the both the row sum vector and column sum vector equal T_n. If n is odd, D_n is unique. If n is even, there are two such matrices (we refer to both of them as diamond ASMs and denote them by D_n) each obtained from the other by reversing the order of the rows. For instance,