

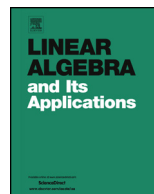


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## Note on the spectral radius of alternating sign matrices



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### ABSTRACT

We show that the  $n \times n$  so-called diamond alternating sign matrix  $D_n$  is the unique  $n \times n$  alternating sign matrix with maximum spectral radius  $\rho_n$ , and that  $\lim_{n \rightarrow \infty} \frac{\rho_n}{n} = \frac{2}{\pi}$ .

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An *alternating sign matrix*, abbreviated to ASM, is an  $n \times n$   $(0, +1, -1)$ -matrix such that, ignoring 0s, the  $+1$ s and  $-1$ s alternate in each row and column beginning and ending with a  $+1$ . In particular, all row and column sums of an ASM equal 1. For example, the matrix

$$\begin{bmatrix} 0 & 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & -1 & +1 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix} \tag{1}$$

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is an ASM. Every permutation matrix is an ASM, and so ASMs can be thought of as generalizations of permutation matrices. A row and column permutation of an ASM, even a simultaneous row and column permutation, need not result in an ASM. For instance, the matrix obtained by interchanging rows 1 and 2 and columns 1 and 2 of the ASM in (1) is not an ASM. We refer the reader to [2] for the history of ASMs and to [1] for a recent combinatorial investigation of ASMs.

Let  $\mathcal{A}_n$  denote the set of all  $n \times n$  ASMs. Let  $\rho(X)$  denote the spectral radius of the  $n \times n$  matrix  $X$ , that is, the maximum absolute value of an eigenvalue of  $X$ . Since all row and column sums of an ASM equal 1, it follows that 1 is an eigenvalue of every ASM with corresponding eigenvector equal to a vector of all 1s. Thus  $\rho(A) \geq 1$  for every ASM; equality occurs for ASMs that are permutation matrices but may occur for other ASMs as well. For instance, let

$$A = \begin{bmatrix} 0 & 0 & +1 & 0 & 0 \\ +1 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of  $A$  is easily computed to be

$$(\lambda - 1)^2(\lambda + 1)(\lambda^2 - \lambda + 1)$$

and hence  $A$  has eigenvalues

$$1, 1, -1, \frac{1 \pm \sqrt{-3}}{2}$$

all of which have absolute value equal to 1. Therefore  $\rho(A) = 1$ . This also follows from the fact that  $A^6 = I_6$  (in fact, all powers of  $A$  are  $(0, +1, -1)$ -matrices).

The purpose of this note is to determine the maximum spectral radius of an  $n \times n$  ASM:

$$\rho_n = \{ \max \rho(A) : A \in \mathcal{A}_n \}.$$

Although every ASM different from a permutation matrix has at least one negative entry, we nonetheless are able to make use of the well-known Perron–Frobenius theory of nonnegative matrices, abbreviated to (PF), (see e.g. Chapter 8 of [3]). We also use the fact that for a square matrix  $B$ ,  $\rho(B^2) = \rho(B)^2$ , and we make use of the following theorem (actually the technique of its proof) of Schwarz [4], which is a consequence of this theory.

**Theorem 1.** *Let  $c_1, c_2, \dots, c_{n^2}$  be  $n^2$  nonnegative numbers. Let  $\mathcal{C}_n$  be the set of all  $n \times n$  matrices whose entries are these  $n^2$  numbers in some order. Then the maximum spectral radius of a matrix in  $\mathcal{C}_n$  is attained by a matrix  $A = [a_{ij}]$  in  $\mathcal{C}_n$  for which the entries in each row and in each column are nonincreasing:*

$$a_{i1} \geq a_{i2} \geq \dots \geq a_{in} \quad (1 \leq i \leq n), \quad \text{and} \quad a_{1j} \geq a_{2j} \geq \dots \geq a_{nj} \quad (1 \leq j \leq n).$$

In replacing in Theorem 1 the maximum spectral radius with minimum spectral radius and non-increasing with nondecreasing, an equally valid theorem results.

If  $X$  is an  $n \times n$  real matrix, then we denote by  $|X|$  the  $n \times n$  nonnegative matrix obtained from  $X$  by replacing each entry with its absolute value. If  $A$  is an  $n \times n$  ASM, then  $|A|$  is a  $(0, 1)$ -matrix. Counting the number of 1s in each row of  $|A|$  we get the row sum vector  $R_n = (r_1, r_2, \dots, r_n)$  of  $|A|$  where each  $r_i$  is necessarily odd. The alternating property of rows of an ASM implies that the  $R_n \leq T_n$  entrywise where  $T_n = (1, 3, 5, \dots, 5, 3, 1)$ . Similarly, we have a column sum vector  $S_n$  of  $|A|$  with all odd components satisfying  $S_n \leq T_n$  entrywise. The diamond ASM  $D_n$  is the  $n \times n$  ASM for which the both the row sum vector and column sum vector equal  $T_n$ . If  $n$  is odd,  $D_n$  is unique. If  $n$  is even, there are two such matrices (we refer to both of them as diamond ASMs and denote them by  $D_n$ ) each obtained from the other by reversing the order of the rows. For instance,

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