Linear Algebra and its Applications 442 (2014) 145–155

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Reflexive bipartite regular graphs

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ARTICLE INFO

Article history:
Received 29 April 2013
Accepted 17 July 2013
Available online 12 August 2013
Submitted by R. Brualdi

MSC:
05C50
05B05

Keywords:
Adjacency matrix
Bounded eigenvalues
Bipartite regular graphs
Block designs

ABSTRACT

A graph is called reflexive if its second largest eigenvalue does not exceed 2. In this paper, we determine all reflexive bipartite regular graphs. Any bipartite regular graph of degree at most 2 is reflexive as well as its bipartite complement. Apart from them, there is a finite number of resulting graphs.

1. Introduction

All graphs considered are simple and undirected. For a graph G on n vertices its characteristic polynomial P_G is just the characteristic polynomial of its adjacency matrix A $(= A(G))$. The eigenvalues of G are the roots of its characteristic polynomial, and they are usually denoted by $\lambda_1(=\lambda_1(G)) \geqslant \lambda_2(=\lambda_2(G)) \geqslant \cdots \geqslant \lambda_n(=\lambda_n(G))$.

There are many results concerning graphs with small second largest eigenvalue and their applications (not to be listed here, see for example [7] and the references therein). In particular, graphs whose second largest eigenvalue does not exceed 2 are called reflexive. These graphs correspond to sets of vectors in the Lorentz space $\mathbb{R}^{p,1}$ with norm 2 and mutual angles 90° and 120°. They are Lorentzian counterparts of the spherical and Euclidean graphs, which occur in the theory of reflection groups, having direct application to the construction and the classification of such groups [5].

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0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.07.020

Most of the theoretical results on reflexive graphs are related to trees and tree-like graphs (see [9] and the corresponding references), while here we consider reflexive bipartite regular graphs (for short, RBR graphs). In the next section we give some initial results, and recall some of those that are obtained in our previous work [7]. In Section 3 we completely determine all RBR graphs. They are represented in the last section along with some comments and theoretical observations.

We fix some notation and terminology. The complement of a graph G we denote by $\overline{G}$, while the disjoint union of k copies of G we denote by kG. The diameter and girth (the length of a shortest cycle) of G are denoted by $\operatorname{diam}(G)$, and $\operatorname{gr}(G)$, respectively. If G is regular, its degree is denoted by r, and then we usually say that G is r-regular. The set of neighbours of a vertex v is denoted by $N(v)$, and $d(u,v)$ stands for a distance between u and v. Finally, we use I and J to denote the identity, and all one matrices, respectively.

The *bipartite complement* of bipartite graph G with two colour classes U and W is bipartite graph $\overline{\overline{G}}$ with the same colour classes having the edge between U and W exactly where G does not. If G is bipartite r-regular graph on $2n$ vertices, its adjacency matrix will usually be given in the following form

$$A_G = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}. \tag{1}$$

If so, then $\overline{\overline{G}}$ is bipartite $(n-r)$-regular graph with adjacency matrix

$$A_{\overline{\overline{G}}} = \begin{pmatrix} 0 & J-N \\ J-N^T & 0 \end{pmatrix}. \tag{2}$$

It is easy to check (compare [12, Theorem 4.1]) that the following equality holds

$$\frac{P_G(x)}{x^2-r^2} = \frac{P_{\overline{\overline{G}}}(x)}{x^2-(n-r)^2}. \tag{3}$$

For a finite set V, let $\mathcal{B}$ be a collection of subsets of the same size of V. Elements of V and $\mathcal{B}$ are called points and blocks, respectively. A symmetric balanced incomplete block design (for short, SBIBD) with parameters (v,r,μ) is a pair $(V,\mathcal{B})$ with the following properties: $|V| = |\mathcal{B}| = v$, each point is contained in r blocks, and every pair of distinct points is contained in exactly μ blocks.

If $V = \{p_1, p_2, \ldots, p_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_v\}$ then the incidence graph of a block design $(V,\mathcal{B})$ is the graph with v vertices and adjacency matrix $N = (n_{ij})$, where $n_{ij} = 1$ when $p_i \in B_j$, and $n_{ij} = 0$ when $p_i \notin B_j$. It is known that the incidence graph of a SBIBD with parameters (v,r,μ) is a (distance-regular) graph with the spectrum $\pm r$, $\pm\sqrt{r-\mu}^{\,v-1}$ (as usual, the exponent stands for the multiplicity of the eigenvalue), and vice versa, a connected bipartite regular graph with four distinct eigenvalues must be the incidence graph of a SBIBD [3].

2. Preliminary results

Every bipartite regular graph (connected or not) of degree at most 2 is reflexive. These graphs are disjoint unions of complete graphs of order either 1 or 2, or disjoint unions of cycles of even orders. In addition, any disconnected RBR graph has degree at most 2 (since it is disconnected, its second largest eigenvalue is equal to its degree), so we can proceed to determine all connected RBR graphs. The set of all such graphs will be denoted by $\mathcal{R}$. Obviously, any graph in $\mathcal{R}$ must have an even order, and thus from now-on we will assume that any graph considered has $2n$ vertices. We prove the following result.

Proposition 2.1. *Every connected bipartite regular graph with $2n$ vertices whose degree is at least $n-2$ is reflexive.*

Proof. The degree r of a graph G described in the proposition is at most n, and then G is a bipartite complement of some bipartite regular graph of degree at most 2. Using Eq. (3), we get $\lambda_2(G) = \lambda_2(\overline{\overline{G}})$, and the result follows. $\square$