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On matrices associated to directed graphs and applications

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ABSTRACT

This paper deals with the notions of 0-incidence and 1-incidence between edges on a directed graph associated to the line graph of a graph. The Laplacian energy and the signless Laplacian energy are obtained in a new way. From these results a relation between both energies is derived. Moreover, we obtain lower bounds for both the largest Laplacian eigenvalue and the largest signless Laplacian eigenvalue and prove that the latter is strictly greater than the first one.

1. Notation and preliminaries

By an (n,m)-graph $G=(\mathcal{V}(G),\mathcal{E}(G))$, for short $G=(\mathcal{V},\mathcal{E})$, we mean an undirected simple graph on $|\mathcal{V}|=n$ vertices and $|\mathcal{E}|=m$ edges. If $u\in\mathcal{V}$ is an end vertex of $e\in\mathcal{E}$ we say that u incides on e.

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If $e \in \mathcal{E}$ has end vertices u and v we say that u and v are adjacent and we denote this edge by $\{u, v\}$. For $u \in \mathcal{V}$ the set of neighbors of u, denoted by $N_G(u)$, is the set of vertices adjacent to u in G, that is, $N_G(u) = \{v \in \mathcal{V}(G)\colon \{u, v\} \in \mathcal{E}\}$. The cardinality of $N_G(u)$, denoted by d_u, is called the degree of u. The adjacency matrix A of a graph G whose vertex set is $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the square matrix of order n, whose entry a_{ij} is equal to 1 if v_i and v_j are adjacent and 0 otherwise. Let $\mathrm{D}(G)$ be the $n \times n$ diagonal matrix of the vertex degrees of G. The matrices $L(G) = \mathrm{D}(G) - A(G)$ and $Q(G) = \mathrm{D}(G) + A(G)$ are the Laplacian and the signless Laplacian matrices of G respectively, see [5,7–9]. It is well known that the spectra (multiset of eigenvalues) of $L(G)$ and $Q(G)$ coincide if and only if G is a bipartite graph, see [8,9].

The spectrum of an $n \times n$ symmetric matrix M is denoted by $\sigma(M) = \{\lambda_1(M), \ldots, \lambda_n(M)\}$, where this sequence of eigenvalues is given in non-increasing order. Considering an eigenvalue λ of M and an associated eigenvector $\mathbf{u}$, the pair $(\lambda, \mathbf{u})$ is called an eigenpair of M.

The line graph $\mathcal{L}(G)$ is the graph whose vertices correspond to edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common. That is, $\mathcal{V}(\mathcal{L}(G)) = \mathcal{E}(G) = \{e_1, e_2, \ldots, e_m\}$ and $\mathcal{E}(\mathcal{L}(G)) = \{\{e_i, e_j\}\colon\ i \neq j,\ e_i \text{ and } e_j \text{ with a common vertex}\}$.

Let $I(G)$ be the (vertex–edge) incidence matrix of the graph G defined as the $n \times m$ matrix whose (i, j)-entry is 1 if the vertex v_i incides to the edge e_j and 0 otherwise. It is well known (see [12]) that

$$I(G)I(G)^t = A(G) + \mathrm{D}(G) = Q(G), \tag{1}$$

$$I(G)^t I(G) = 2\mathrm{I}_m + A_{\mathcal{L}}(G), \tag{2}$$

where I_m denotes the identity matrix of order m and $A_{\mathcal{L}}(G)$ is the adjacency matrix of $\mathcal{L}(G)$.

Recall that if A and B are any matrices of orders $t \times s$ and $s \times t$, respectively, then AB and BA have the same nonzero eigenvalues. According to Nikiforov (see [16]), for a real symmetric $n \times n$ matrix T its Ky Fan k-norm $\|T\|_{F_k}$ is the sum of the k largest absolute values of its eigenvalues. So, for an (n, m)-graph G its energy, $E(G)$ (a well-studied concept introduced by Gutman in [10]) is given by $\|A(G)\|_{F_n}$. In the same way, as the average of eigenvalues Laplacian and signless Laplacian is $\frac{2m}{n}$, the Laplacian energy (see [13,18]) and the signless Laplacian energy of G (see [11]) are defined by

$$LE(G) = \left\| L(G) - \frac{2m}{n}\mathrm{I}_n \right\|_{F_n} \quad \text{and} \quad LE^+(G) = \left\| Q(G) - \frac{2m}{n}\mathrm{I}_n \right\|_{F_n}, \tag{3}$$

respectively. Evidently, for a bipartite graph both Laplacian and signless Laplacian energy coincide.

From any graph G we can obtain a simple digraph (directed graph) G' by choosing only one ordered pair among (u, v) and (v, u) for every edge $\{u, v\}$ in $\mathcal{E}(G)$. These pairs are called directed edges or arcs. If $e = (u, v)$ is an arc in G' then u and v are the initial and the terminal vertex of e, respectively. Two arcs $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ with a common vertex are called 1-*incident* if either $v_1 = u_2$ or $v_2 = u_1$; otherwise, we say that they are 0-*incident*.

In this paper using the notions of 0-incidence and 1-incidence between edges on a simple digraph we obtain new formulae to compute the Laplacian and the signless Laplacian energies of any graph G. Finally, we compare these graph invariants. Moreover, we obtain lower bounds for both the largest Laplacian eigenvalue and the largest signless Laplacian eigenvalue and prove that the latter is strictly greater than the first one.

2. Incidence matrices for directed graphs

Let $\mathcal{O}(G)$ denote the set of all simple digraphs G' obtained from G. For $G' \in \mathcal{O}(G)$, by using 0-incidence and 1-incidence notions we introduce two graphs, the 0-*incident-line graph* $\mathcal{L}_0(G')$ and the 1-*incident-line graph* $\mathcal{L}_1(G')$ which are defined by

$$\mathcal{V}\big(\mathcal{L}_0\big(G'\big)\big) = \mathcal{V}\big(\mathcal{L}_1\big(G'\big)\big) = \mathcal{V}\big(\mathcal{L}(G)\big) = \mathcal{E}(G),$$