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Hermitian matrices with a bounded number of eigenvalues

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ABSTRACT

Conjugation covariants of matrices are applied to study the real algebraic variety consisting of complex Hermitian matrices with a bounded number of distinct eigenvalues. A minimal generating system of the vanishing ideal of degenerate three by three Hermitian matrices is given, and the structure of the corresponding coordinate ring as a module over the special unitary group is determined. The method applies also for degenerate real symmetric three by three matrices. For arbitrary n partial information on the minimal degree component of the vanishing ideal of the variety of $n \times n$ Hermitian matrices with a bounded number of eigenvalues is obtained, and some known results on sum of squares presentations of subdiscriminants of real symmetric matrices are extended to the case of complex Hermitian matrices.

1. Introduction

Let $\mathbb{F}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. For a matrix $A \in \mathbb{C}^{n\times n}$ denote $\bar{A}$ and A^T the complex conjugate and transpose of A, respectively. Fix a positive integer $n \geqslant 2$, and let $\mathcal{M}$ be one of the following $\mathbb{F}$-subspaces of $\mathbb{C}^{n\times n}$:

(a) the Hermitian matrices $\mathrm{Her}(n) = \{A \in \mathbb{C}^{n\times n} \mid \bar{A} = A^T\}$;
(b) the real symmetric matrices $\mathrm{Sym}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$;
(c) all $n \times n$ complex matrices $\mathbb{C}^{n\times n}$;
(d) the complex symmetric matrices $\mathrm{Sym}(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid A^T = A\}$.

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For $k = 0, 1, \dots, n-1$ consider the following subset of $\mathcal{M}$:

$$\mathcal{M}_k := \{A \in \mathcal{M} \mid \deg(m_A) \leqslant n-k\}$$

where m_A stands for the minimal polynomial of the matrix A. Clearly $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_k \supset \mathcal{M}_{k+1}$, and for a fixed $k \in \{0, 1, \dots, n-1\}$ we have the inclusions

$$\begin{array}{ccc} (\mathbb{C}^{n\times n})_k & \supset & \mathrm{Her}(n)_k \\ \cup & & \cup \\ \mathrm{Sym}(n,\mathbb{C})_k & \supset & \mathrm{Sym}(n,\mathbb{R})_k \end{array}$$

We have also the equalities $\mathrm{Her}(n)_k = (\mathbb{C}^{n\times n})_k \cap \mathrm{Her}(n)$ and $\mathrm{Sym}(n,\mathbb{R})_k = \mathrm{Sym}(n,\mathbb{C})_k \cap \mathrm{Sym}(n,\mathbb{R}) = \mathrm{Her}(n)_k \cap \mathbb{R}^{n\times n}$. Obviously $\mathcal{M}_k$ is the common zero locus in $\mathcal{M}$ of the coordinate functions of the polynomial map

$$\mathcal{P}_k : \mathcal{M} \to \bigwedge^{n-k+1} \mathcal{M}, \quad A \mapsto I_n \wedge A \wedge A^2 \wedge \cdots \wedge A^{n-k}$$

where I_n is the $n \times n$ identity matrix and $\bigwedge^l \mathcal{M}$ is the lth exterior power of $\mathcal{M}$. In particular, $\mathcal{M}_k$ is an affine algebraic subvariety of the affine space $\mathcal{M}$, and it is natural to raise the following question:

Question 1.1. *Do the coordinate functions of the polynomial map $\mathcal{P}_k$ generate the vanishing ideal $\mathcal{I}(\mathcal{M}_k)$ in $\mathbb{F}[\mathcal{M}]$ of the affine algebraic subvariety $\mathcal{M}_k \subset \mathcal{M}$?*

Above $\mathbb{F}[\mathcal{M}]$ is the coordinate ring of $\mathcal{M}$, so $\mathbb{F} = \mathbb{R}$ in cases (a), (b) whereas $\mathbb{F} = \mathbb{C}$ in cases (c), (d), and $\mathbb{F}[\mathcal{M}]$ is a polynomial ring over $\mathbb{F}$ in $\dim_{\mathbb{F}}(\mathcal{M})$ variables. Recall that the *vanishing ideal* of $\mathcal{M}_k$ is

$$\mathcal{I}(\mathcal{M}_k) := \{f \in \mathbb{F}[\mathcal{M}] \mid f|_{\mathcal{M}_k} \equiv 0\} \triangleleft \mathbb{F}[\mathcal{M}]$$

We have $\mathcal{M}_0 = \mathcal{M}$, so $\mathcal{I}(\mathcal{M}_0)$ is the zero ideal, and $\mathcal{P}_0$ is the zero map. From now on we focus on $\mathcal{M}_{k+1}$ and $\mathcal{I}(\mathcal{M}_{k+1})$ where $k = 0, 1, \dots, n-2$.

Our original interest was in the real cases (a) and (b): then $\mathbb{F} = \mathbb{R}$ and all $A \in \mathcal{M}$ are diagonalizable with real eigenvalues, hence

$$\mathcal{M}_{k+1} = \{A \in \mathcal{M} \mid A \text{ has at most } n-k-1 \text{ distinct eigenvalues}\} \tag{1}$$

It follows from (1) that in the real cases $\mathcal{M}_{k+1}$ (for $k = 0, 1, \dots, n-2$) is the zero locus of a single polynomial $\mathrm{sDisc}_k \in \mathbb{R}[\mathcal{M}]$, defined by

$$\mathrm{sDisc}_k(A) := \sum_{1 \leqslant i_1 < \cdots < i_{n-k} \leqslant n} \prod_{1 \leqslant s < t \leqslant n-k} (\lambda_{i_s} - \lambda_{i_t})^2$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Note that $\mathrm{sDisc}_k(A)$ coincides with the *k-subdiscriminant* of the characteristic polynomial of A (we refer to Chapter 4 of [1] for basic properties of subdiscriminants), and sDisc_k is a homogeneous polynomial function on $\mathcal{M}$ of degree $(n-k)(n-k-1)$. In the special case $k = 0$ we recover the *discriminant* $\mathrm{Disc} = \mathrm{sDisc}_0$. The ideal $\mathcal{I}(\mathcal{M}_{k+1})$ is generated by homogeneous elements (with respect to the standard grading on the polynomial ring $\mathbb{F}[\mathcal{M}] = \bigoplus_{d=0}^{\infty} \mathbb{F}[\mathcal{M}]_d$). In [6] it was deduced from the Kleitman–Lovász theorem (cf. Theorem 2.4 in [18]) that $\frac{1}{2}\deg(\mathrm{sDisc}_k) = \binom{n-k}{2}$ is the minimal degree of a non-zero homogeneous component of $\mathcal{I}(\mathcal{M}_{k+1}) = \bigoplus_{d=0}^{\infty} \mathcal{I}(\mathcal{M}_k)_d$ (in fact [6] deals with the case $\mathcal{M} = \mathrm{Sym}(n,\mathbb{R})$ only, but the proof of Corollary 5.3 in [6] works also for the case $\mathcal{M} = \mathrm{Her}(n)$, see Proposition 7.2 and Theorem 8.1 (i) in the present paper). Since the polynomial map $\mathcal{P}_{k+1}$ is homogeneous of degree $\binom{n-k}{2}$, its coordinate functions are contained in the homogeneous component $\mathcal{I}(\mathcal{M}_{k+1})_{\binom{n-k}{2}}$. So an affirmative answer to Question 1.1 would imply in particular that $\mathcal{I}(\mathcal{M}_{k+1})$ is generated by its minimal degree non-zero homogeneous component.