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Polynomial identities for the Jordan algebra of a degenerate symmetric bilinear form

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ABSTRACT

Let J_n be the Jordan algebra of a degenerate symmetric bilinear form. In the first section we classify all possible G-gradings on J_n where G is any group, while in the second part we restrict our attention to a degenerate symmetric bilinear form of rank $n-1$, where n is the dimension of the vector space V defining J_n. We prove that in this case the algebra J_n is PI-equivalent to the Jordan algebra of a nondegenerate bilinear form.

1. Introduction

Let A be an associative algebra over a field F of characteristic 0, and denote by $Id(A)$ its T-ideal of identities. Since $\operatorname{char} F = 0$ it suffices to study only the multilinear polynomial identities of A and let P_n be the vector subspace of the free associative algebra $F(X)$ of multilinear polynomials in $x_1, \ldots, x_n$. We assume that the set X of the free generators is countable and infinite. Thus in order to study the identities of A one studies the intersections $P_n \cap Id(A)$, $n \geqslant 1$. But for practical purposes these intersections are not suitable since they tend to become very large as $n \to \infty$. Therefore one is led to study the quotients $P_n(A) = P_n/(P_n \cap Id(A))$. The dimension $c_n = c_n(A) = \dim P_n(A)$ is called the n-th *codimension* of A; the sequence of codimensions for a given algebra is one of the

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most important characteristics of the identities of A. In [4,6] Giambruno and Zaicev proved that the sequence $(c_n(A))^{1/n}$ converges, and its limit is always an integer, called the PI-*exponent* of A. Since then an extensive research on the exponent of PI algebras has been conducted. It is of interest to study the minimal algebras with respect to their PI-exponent. Recall that A is minimal when for every algebra B such that $Id(A) \subset Id(B)$ (a proper inclusion), the PI-exponent of B is less than that of A. The interested reader may wish to consult Chapters 7 and 8 of the monograph [5] for further reading about minimal algebras and varieties.

One may define, and study analogous concepts for large classes of nonassociative algebras as well. Here we mention only that the PI-exponent of a nonassociative algebra need not be an integer.

In the case of Jordan algebras, one of the nontrivial cases where the identities are known is that of the algebras B_n and B of a nondegenerate symmetric bilinear form, to be defined below. These results are due to Vasilovsky [12]. Recall that earlier Iltyakov [7] developed methods to study the identities in these algebras and proved that the variety generated by B_n is Spechtian. A latter result appeared in [10] describing the ordinary and $\mathbb{Z}_2$-graded polynomial identities for $UJ_2(F)$, i.e. the Jordan algebra of 2×2 upper triangular matrices. This algebra is a particular case of the algebra studied in this paper, because it is isomorphic to B_2 with a degenerate bilinear form of rank 1. Apart from the results mentioned above very little is known about the concrete form of the identities satisfied by a given algebra.

The importance of B_n in the class of Jordan algebras, is highlighted by a well-known result stating that if F is algebraically closed, then any finite dimensional simple Jordan algebra, up to isomorphism, is one of the following: $M_n(F)^+$, the special Jordan algebra of $n \times n$ matrices, $M_n(F)^t$, the algebra of $n \times n$ symmetric matrices with respect to the transpose involution, $M_{2n}(F)^s$, the $2n \times 2n$ symmetric matrices with respect to the symplectic involution, and the algebra B_n when the form is nondegenerate.

Group gradings on algebras and the corresponding graded identities have become an area of extensive study. We refer the interested reader to the survey [2] for further reading and reference (see also [9]) concerning gradings and graded identities.

In the first part of this paper we generalize the result given by Bahturin and Shestakov which classifies all possible G-gradings on B_n when the bilinear form is nondegenerate and G is any group (see [1]). In our case b becomes degenerate. The second part is devoted to the computation of a basis of the T-ideal of ordinary polynomial identities of B_n, generalizing a theorem due to Vasilovsky in [12] where the same T-ideal was computed in the case of the form is nondegenerate.

2. Preliminaries

All algebras and vector spaces we consider will be over a fixed field F of characteristic 0. Any additional restrictions on the field will be mentioned explicitly.

Let A be an associative algebra and denote by A^+ the vector space of A equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Then A^+ is a Jordan algebra. The Jordan algebras of this type as well as their subalgebras are called *special*, otherwise they are called *exceptional*. Let V be a vector space equipped with a symmetric bilinear form b, and let $B = F \oplus V$. Define a multiplication $\circ$ on B by $(\alpha + u) \circ (\beta + v) = (\alpha\beta + b(u, v)) + (\alpha v + \beta u)$, $\alpha, \beta \in F$, $u, v \in V$. Then B is a Jordan algebra. If $\dim V = n$ we shall denote it by B_n. (Clearly these algebras depend on the form b.) In order to simplify the notation, for the rest of the paper we shall denote by B_n the Jordan algebra equipped with a nondegenerate bilinear form, and by $J_n = B_m \oplus D_k$ the Jordan algebra with a degenerate bilinear form of rank m, where D_k is the vector space spanned by the degenerate elements of the basis of V.

If A is an algebra we denote the *associator* of $a, b, c \in A$ as $(a, b, c) = (ab)c - a(bc)$; here ab is the product in A.

Let G be a group and A an algebra. We say that A is G-graded if $A = \bigoplus_{g \in G} A_g$ is a direct sum of vector subspaces such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. The elements of A_g are homogeneous of degree g. The homogeneous degree of an $a \in A_g$ is denoted by $|a|$ or $\deg a$.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite countable set. We denote by $F(X)$ and by $J(X)$ the free (unitary) associative and the free Jordan algebra freely generated by X over F, respectively. A polynomial $f = f(x_1, \ldots, x_n) \in F(X)$ is a *polynomial identity* (a PI or an identity) for the associative algebra A if