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On minors of the compound matrix of a Laplacian

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ABSTRACT

Let L be an $n \times n$ matrix with zero row and column sums, $n \geqslant 3$. We obtain a formula for any minor of the $(n-2)$-th compound of L. An application to counting spanning trees extending a given forest is given.

1. Introduction

We consider graphs with no loops but possibly with multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We usually take $V(G) = \{1, \dots, n\}$. Let G be a graph with $V(G) = \{1, \dots, n\}$. The adjacency matrix A of G is the $n \times n$ matrix with a_{ij} equal to the number of edges between i and j if $i \neq j$, and $a_{ii} = 0$, $i = 1, \dots, n$. The Laplacian matrix L of G is defined as $D - A$, where D is the diagonal matrix of vertex degrees. Clearly, L is symmetric with zero row and column sums. (A row sum of a matrix is the sum of all the elements in a row. A column sum is defined similarly.) It is well-known that G is connected if and only if the rank of L is $n-1$. We refer to [1] for background concerning graphs and matrices.

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The Matrix-Tree theorem asserts that any cofactor of the Laplacian equals the number of spanning trees of the graph. A combinatorial interpretation of all minors of the Laplacian matrix can also be given, see [2,3].

We introduce some further notation and definitions. The determinant of the square matrix A is denoted by $|A|$. Let A be an $m \times n$ matrix and let $1 \leqslant k \leqslant \min\{m, n\}$. We denote by $Q_{k,n}$, the set of increasing sequences of k elements from $\{1, \dots, n\}$. For indices $I \subset \{1, \dots, m\}$, $J \subset \{1, \dots, n\}$, $A[I|J]$ will denote the submatrix of A formed by the rows indexed by I and the columns indexed by J. The k-th *compound* of A, denoted by $C_k(A)$, is an $\binom{m}{k} \times \binom{n}{k}$ matrix defined as follows. The rows and the columns of $C_k(A)$ are indexed by $Q_{k,m}$, $Q_{k,n}$, respectively, where the ordering is arbitrary but fixed. If $I \in Q_{k,m}$, $J \in Q_{k,n}$, then the (I, J)-entry of $C_k(A)$ is set to be $|A[I|J]|$.

If A and B are matrices of order $m \times n$, $n \times m$ respectively and if $1 \leqslant k \leqslant \min\{m, n, p\}$, then it follows from the Cauchy–Binet formula that $C_k(AB) = C_k(A)C_k(B)$.

If $S, T \subset \{1, \dots, n\}$, then we denote by $A(S|T)$, the submatrix of A obtained by deleting the rows indexed by S and the columns indexed by T. If $S = \{i\}$ and $T = \{j\}$, then we denote $A(S|T)$ simply by $A(i|j)$. Similarly, if $S = \{i, j\}$ and $T = \{k, \ell\}$, then we denote $A(S|T)$ by $A(ij|k\ell)$ and so on. The notation $A(S|:)$ will be used to denote the matrix formed by deleting the rows corresponding to S (and keeping all the columns); $A(:|T)$ is defined similarly.

Let K_n be the complete graph on the vertices $\{1, \dots, n\}$. We assume $n \geqslant 3$. Let $E(K_n)$ be the set of edges of K_n, which evidently is the set of unordered pairs of elements in $\{1, \dots, n\}$. If G is a graph with vertex set $\{1, \dots, n\}$ and if vertices i and j are adjacent, then we denote the edge joining i and j by (ij). We will be interested in $C_{n-2}(L)$ for a matrix L with zero row and column sums. (We do not impose any further conditions such as symmetry or the off-diagonal elements being non-positive.) The elements of $C_{n-2}(L)$ are indexed by subsets of $\{1, \dots, n\}$ of cardinality $n-2$. We prefer to index them instead by unordered pairs of elements from $\{1, \dots, n\}$. Thus if $e = (ij)$ and $f = (k\ell)$ are in $E(K_n)$, then the (e, f)-entry of $C_{n-2}(L)$ is given by $|L(ij|k\ell)|$. The objective of the present paper is to provide a formula for any minor of $C_{n-2}(L)$. The motivation for our work is the paper by Burton and Pemantle [4], where a formula for a principal minor of $C_{n-2}(L)$ is given, in the special case when L is the Laplacian matrix of a graph. This result will be stated in Section 3. In Section 2 we prove several preliminary results and then obtain our main result.

2. Results

The following result is well-known. We include a proof for completeness.

Lemma 1. *Let $L = ((\ell_{ij}))$ be an $n \times n$ matrix with zero row and column sums. Then the cofactors of L are all equal.*

Proof. In the matrix $L(1|1)$, add all the columns to its first column. Then since L has zero row sums, the first column now becomes the negative of the first column of $L(1|2)$. Thus $|L(1|1)| = -|L(1|2)|$ and hence the cofactors of ℓ_{11} and ℓ_{12} are equal. We can prove similarly that all the cofactors of L are equal. □

If L is an $n \times n$ matrix with zero row and column sums then we denote the common value of its cofactors by $\tau(L)$. Note that $C_{n-1}(L)$ has each element $\pm\tau(L)$. It turns out that there are intricate relationships among the $(n-2) \times (n-2)$ minors of such a matrix L. We begin by observing some such relationships in the next two results and these will form one of our main tools.

Lemma 2. *Let L be an $n \times n$ matrix with zero column sums. Let $1 \leqslant i < j < k \leqslant n$ and $1 \leqslant u < v \leqslant n$. Then*

$$\left|L(ij|uv)\right| = (-1)^{k-i-1}\left|L(jk|uv)\right| + (-1)^{k-j}\left|L(ik|uv)\right|. \tag{1}$$

Proof. In the matrix $L(ij|uv)$, add all the rows to the k-th row (i.e., the row indexed by k). Let the resulting matrix be X. Since the column sums of L are zero, the row k of X equals the negative of